THE ALTERNATIVE METHOD AND BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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JUNE, 1982

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A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

by
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to the
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

JUNE, 1982

CERTIFICATE

Certified that the thesis entitled ''The Alternative Method And Boundary Value Problems For Functional Differential Equations'' by B.K.Mahapatra has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

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ACKNOWLEDGEMENT

I take this opportunity to express my sincere thanks to my supervisor Professor Dr. P.C. Das for his invaluable assistance throughout the research programme. I also place to record my deep sense of gratitude to Professor Dr. Swadhin Pattnayak for encouraging me to go in for research. For useful discussions and otherwise, I would like to thank my friends Dr. H.S. Sharatchandra and Dr. M. Venkatesulu. Finally, for painstaking typing I thank Mr. Ashok Kumar Bhatia.

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INTRODUCTION

In the last several years, functional differential equations (FDE) have found widespread applications in several areas of science, engineering and social sciences like economics. There is indeed a large literature already existing in this area with several well-known texts and monographs-Hale [36], Halanay [30]. Driver [26], Bellman and Cooke [1] etc. Just as in the theory of differential equations, for FDE also initial value problems, boundary value problems and qualitative theory form major areas of research. The boundary value problems for FDE and in particular for delay differential equations had been the subject of study of many. See Halanay [30]. The problem of periodic solutions had been even more intensively studied by Perello [53], Halanay [31], Hale [33], Mawhin [48].

There are several methods of dealing with boundary value problems, mostly based on fixed point theory, Leray-Schauder degree, monotone operator theory besides some particular methods. Another approach under the name of alternative method due to Lyapunov-Schmidt [46,55] put into a functional analytic framework by Cesari [8] has been recently very popular and has found wide applications in several types of boundary value problems. Its greatest merit lies in its capacity to deal with problems at resonance-See Cesari [13,14]. The basic features of this method as developed in the works of Cesari [8,9,11-14],

Knoblock [42], Locker [44,45], Hale, Bancroft and Sweet [34], Cesari and Kannan [10,15], Osborne and Sather [52] and others **is** as follows. (A detailed bibliography of related results is available in Cesari [16]).

Let $Lx = Nx$, $x \in D(L) \cap D(N)$ denote an operator equation. The main idea is to split the given problem by use of projections and partial inverse maps into a system of two different equations possibly in different spaces, one of which is called the auxiliary equation, and the other the bifurcation equation. Whenever the auxiliary equation can be solved by the general theorems of functional analysis, the original problem essentially reduces to solving an operator equation in a finite dimensional space. The latter is more amenable to solution by algebraic methods or Brouwer degree for existence purposes under certain solvability conditions.

The motivation for this **thesis arose** from the possibility of developing or applying as such the alternative method to FDE. This we do when we solve the Van der Pol equation in Chapter 2 and the single species model for the population growth in the third chapter. Earlier several attempts have been made to use the method of alternative in some form or other to FDE.

In his paper, Perello [53] considers functional differential equations of the type

$$x'(t) = L(x_t) + N(t, x_t, \mu) \tag{0.1}$$

where $L(\emptyset)$ is linear in $\emptyset$ and $N(t,\emptyset,\mu)$ is a perturbation term which for small $\emptyset$ and μ is small relative to the linear part. If $N(t,\emptyset,\mu)$ is periodic in t, Perello gives the necessary and sufficient alternative formulation for the existence of a T-periodic solution of (0.1) which in the first approximation reduces to a T-periodic solution of the linear equation. The theory represents an extension to delay equations in function space setting of the method used by Cesari and Hale for ordinary differential equations. The problem associated with this method is the nonavailability of the form of the partial inverse operator for the operator L.

In his paper, Mawhin [48] also studies the existence of periodic solutions of nonlinear FDE. He uses the method of alternative where L consists of the highest order ordinary differential operator. In the application of the method of alternative to FDE, Das and Venkatesulu [23] took L to be the linear part of the ordinary differential operator. In Mawhin, the linear part consists of a simple first order differential operator whereas Das and Venkatesulu allowed a more general form of differential operator.

In his book on population dynamics, Volterra [57] proposes a system of equations which accounts for possible lag effects in the interaction of the species. This he does by considering the system

$$N_i' = N_i(b_i - d_{ii}\int_0^\infty N_i(t-s)k_{ij}(s)ds + d_{ij}\int_0^\infty N_j(t-s)k_{ij}(s)ds)$$

for $1 \leq i,j \leq 2$, $i \neq j$ where $N_i = N_i(t)$ denotes the population size of the i^{th} species at time t and $k_{ij}(\cdot)$ denote the interaction coefficient measuring the effect of the j^{th} species upon the growth rate of the i^{th} species.

A delay sometimes causes oscillatory behavior. These oscillations can be either sustained or otherwise depending on the significance of the delay compared to other parameters in the model. The latter possibility of sustained, periodic oscillations around an equilibrium has been studied by several authors beginning with the early work of Wright [59], Kakutani and Markus [40], on the single lag logistic model and continuing with the more recent work of Nussbaum [51].

In his paper, Cushing [21] demonstrates the existence of non-constant periodic solutions of the general single-species model of the type

$$x'(t) = -\lambda_1 x(t) - \lambda_2 \int_0^\infty x(t-s)dh(s) + g(x,\lambda_1,\lambda_2)(t)$$

where λ_1 and λ_2 are parameters.

In the present thesis, as an example for the theory developed in Chapter 3, we take up this equation and demonstrate the existence of a periodic solution via the alternative method.

Mostly in the theory of FDE, one writes the solution depending on the initial function through an evolution operator and under various conditions sets out to prove that this operator satisfies the boundary conditions [28,29,53,56]. This way

one gets a solution which is continuous at the initial point. It is all the same possible to approach these equations as the usual ordinary differential equations where the right hand side function is to be suitably determined in an initial interval in such a way that the functional boundary conditions are satisfied by the solution. This approach was adopted in a paper by Das and Venkatesulu [23] dealing with periodic solutions of FDE.

Within this general framework, the method adopted in proving the existence of solutions to the boundary value problem is the method of alternative which has evolved to be a useful tool in recent years. Most of the results where alternative method was used for solutions of boundary value problems have dealt with linear boundary conditions except [25] where a slight deviation from the standard formulation of the abstract problem from the concrete boundary value problem turned out to be crucial. The variation of the same approach is adopted in the first chapter of the present thesis in order to treat nonlinear boundary conditions.

We have taken an equation of the type

$$\tau x \equiv x' - A(t)x = X(t, x_t).$$

$$Zx \equiv Mx_a - Bx_b.$$

over the interval $J = [\, a, b \,]$ where $A(t)$ is an nxn matrix function composed of C^2 functions on J. $M : D(M) \subseteq S_1 \rightarrow R(M) \subseteq S_1$,

is a densely defined closed linear operator with closed range, $S_1 = L_n^2[-\nu, 0]$; $B : D(B) \subseteq S_1 \to R(B) \subseteq S_1$ is an operator not necessarily linear ; X is a nonlinear functional. We denote by L the operator $Lx \equiv \ell x$ and by H, the partial inverse operator for L. We define certain projection operators P_{m_1} and Q_{m_1} and study their relations with L and H. Similarly, we define projections P'_{m_1} and Q'_{m_1} and study their relations with M and F (F being the partial inverse for the boundary operator M). We convert the given equation into a fixed point problem for the operator T given by

$$Tx = x + H(I-Q_m)G_{g^*}(x).$$

We use the contraction mapping principle to get a fixed point of T and the degree theory to solve the bifurcation equation.

As an example for the theory developed in Chapter One, we have proved in Chapter Two, the existence of a solution to the following Van der Pol equation with periodic boundary conditions

$$y'' + \alpha(1-y^2)y' = \beta(y(t-1/4) + \sin 2\pi t) \quad |\alpha| \leq 0.1, \ |\beta| \leq 0.1$$

$$y(0) = y(1) \qquad\qquad y'(0) = y'(1).$$

In the third chapter we consider abstract, operational equations of the form $Lx = Nx$ and $Lx + \alpha Ax = Nx$ where $L : D(L) \subseteq X \to Y$, $N : D(N) \subseteq X \to Y$, $A : X \to Y$ are operators, L linear, not necessarily bounded, N and A continuous not necessarily linear, X and Y Banach spaces and α a real

parameter. We prove in terms of the alternative method and the Leray Schauder topological degree that under certain assumptions on N and A, there exist solutions to $Lx = Nx$ and $Lx + \alpha Ax = Nx$ for $|\alpha| \leq \alpha_0$ for some $\alpha_0 < 0$ and moreover the solutions remain uniformly bounded as α is allowed to pass through the point of resonance $\alpha = 0$. The conditions are easily comparable with the conditions imposed by Cesari.

We have defined as in the previous chapter, projections, P,Q and partial inverse operator H satisfying the usual Cesari-type conditions. In the first theorem we consider the case $Lx = Nx$ when $||Nx|| \leq J_0$. In the second theorem, N is allowed to have a limited growth. The third and fourth theorems deal with the equation $Lx + \alpha Ax = Nx$ again when N is bounded and when N has limited growth. In each of these cases we convert the given equation into a fixed point problem and apply the Leray-Schauder degree theory.

In the final chapter, we give an alternative approach to the problem of periodicity of solution of a class of FDE by considering a domain space which has some extra property other than the ones usually demanded in the literature. This makes the FDE amenable to some of the methods developed earlier.

CHAPTER - 1

Several authors starting with Perello [53] have used the method of alternative for establishing the existence of solutions of boundary value problems of FDE. But the function space approach through the evolution operators naturally meets with difficulties in getting the partial inverse of L. So Mawhin [48] and Das and Venkatesulu [23] have tried to treat boundary value problem for FDE in the particular case of periodic boundary conditions as simply ordinary differential equations where the right hand side function is defined through suitable extension of the left of the initial interval. In most of the literature the boundary conditions are taken as linear except Mawhin [48] and Venkatesulu [59]. Here we deal with FDE with nonlinear boundary conditions.

But in the present work unlike the above authors, we introduce another set of projection operators for the boundary operators in order to deal with the nonlinearity of the boundary conditions. In the case of boundary value problem for FDE this helps us to define the extension of the operator in a suitable way. Before stating the problem precisely we introduce some notations.

1. Notations :

Let J denote a closed interval $[a, b]$ where $b > a$. $J_1 = [-\nu, 0]$ where $0 < \nu < b-a$. For a function y defined

on the real line $y^{(m)}$ denotes the m^{th} derivative of y. $C^m(J)$ denotes the space of all m times continuously differentiable real valued functions on J and $C_n^m(J)$ denotes the space of all functions $x = (x_1, \ldots, x_n)$ represented through a column vector with components belonging to $C^m(J)$. For any function x on $[a-\nu, b]$, the function x_t is defined by $x_t(\theta) = x(t+\theta)$, $\theta \in J_1$ for $t \in J$. $L^2[a,b]$ denotes the real Hilbert space of all square integrable real-valued functions on J. $S = L_n^2[a,b]$, the real Hilbert space of all square integrable functions $x = (x_1, x_2, \ldots, x_n)$ on J each of whose components belongs to $L^2[a,b]$.

The usual inner product and norm in S are denoted through $(.,.)$ and $\| . \|$ respectively. I denotes the identity operator on S. $\tilde{S}$ denotes set of all functions belonging to S which are essentially bounded. The function μ on $\tilde{S}$ is defined by $\mu(x) = \max_{i=1,2,\ldots n} \text{ess sup} |x_i|$, $x \in \tilde{S}$. Similarly $S_1 = L_n^2[-\nu, 0]$ and $(.)_1$ and $\| . \|_1$ denote the inner product and norm respectively in S_1. I_1 denotes the identity operator on S_1. $\tilde{S}_1$ denotes the set of all functions in S_1 which are essentially bounded. The function μ_1 on $\tilde{S}_1$ is defined by $\mu_1(g) = \max_{i=1,\ldots,n} \text{ess sup} |g_i|$, $g \in \tilde{S}_1$. H(J) denotes the space of all absolutely continuous functions x on J with derivative $x^{(1)} \in S$. $H_o(J)$ denotes the space of all functions of H(J) vanishing at both the end points $t = a$ and $t = b$. For an operator T, D(T), N(T), and R(T) denote the domain,

the null space and the range of T, respectively. For an operator T, T^* denotes the adjoint of T provided the adjoint exists. $\langle w_1,\dots,w_m\rangle$ denotes the linear span of the vectors $w_1,\dots,w_m$. For a set $E \subseteq S$, $E^\perp$ denotes the orthogonal complement of E in S. Finally, R^n denotes the n-dimensional real space with the usual Euclidean norm $|\cdot|$.

2. Formulation of the problem :

We consider the following system of n first order nonlinear functional differential equations with sufficiently general boundary conditions.

$$\tau x \equiv x^{(1)} - A(t)x = X(t, x_t) \qquad (1.1)$$

$$Zx \equiv Mx_a - Bx_b = 0. \qquad (1.2)$$

over the interval $J = [a,b]$, where $A(t)$ is an nxn matrix function composed of C^2 functions on J. We assume the following.

(i) $X(t,g)$ [$= (X_1(t,g),\dots,X_n(t,g))$ - column vector] is defined for $t \in J$ and $g \in S_1$ with $\mu_1(g) \leq R$ where $R > 0$ is a real number.

(ii) $X(\cdot,g) \in S$ for each fixed g.

(iii) There exist real numbers $k_i \geq 0$ $(i = 1,2,\dots,n)$ such that

$$|X_i(t,g^*) - X_i(t,g^{**})| \leq K_i\Big(\sum_{j=1}^{p}\sum_{i=1}^{n} |g_i^*(\theta_j)-g_i^{**}(\theta_j)|^2\Big)^{1/2},$$
$$t \in J \qquad (1.3)$$

where $\theta_1, \theta_2, \ldots, \theta_j \in J_1$ and $g = (g_1, g_2, \ldots, g_n)$- column vector.

(iv) $M : D(M) \subsetneq S_1 \to R(M) \subsetneq S_1$ is a densely defined closed linear operator with closed range. Null spaces of M and M^* are of finite dimensions and the dimension of null space of M is greater than (or equal to) the dimension of null space of M^*.

(v) $B : D(B) \subseteq S_1 \to R(B) \subsetneq S_1$ is an operator (not necessarily linear) with domain $D(B)$ of N defined by $D(B) = \{g \in \tilde{S}_1 \mid \mu_1(g) \leq R_1\}$ and $R(B)$ $\tilde{S}_1$.

(v) There exists constant $K \geq 0$ such that

$$\| Bg^* - Bg^{**} \|_1 \leq K \| g^* - g^{**} \|_1. \tag{1.4}$$

With the above assumptions, we shall establish the existence of a solution to the problem (1.1)-(1.2).

3. Operator L and some related properties :

We define the operator $L : D(L) \subseteq S \to R(L) \subsetneq S$ by

$$D(L) = H(J), \; Lx = \tau x. \tag{1.5}$$

L possesses the following well-known properties :

(i) $D(L)$ is dense in S.

(ii) L is a closed linear operator

(iii) $R(L)$ is closed in S. In fact $R(L) = S$.

(iv) $\dim N(L) = n$ and $\dim N(L^*) = 0$.

Proofs of (i), (ii) and (iv) are well known and we can refer to [3]. Proof of (iii) follows from lemma 1.2, to appear.

Choose functions $\phi_1, \phi_2, \ldots, \phi_n \in C_n^{\sim}(J)$ to form an orthonormal basis for $N(L)$ (See [17] or [35]). We observe that the restriction of L to the subspace $D(L) \cap N(L)^{\perp}$ is a 1-1 closed linear operator with closed range. Hence by the Closed Graph Theorem, the inverse map $H = [L|_{D(L) \cap N(L)^{\perp}}]^{-1}$ is a 1-1 continuous linear operator with domain S, range $D(L) \cap N(L)^{\perp}$, and

$$LHy = y \text{ for all } y \in R(L) = S$$

and
$$HLx = x - \sum_{i=1}^{n} (x, \phi_i)\phi_i \text{ for all } x \in D(L). \tag{1.6}$$

We note that H is a continuous right inverse of L. We next define projections P_m and Q_m and study their relations with L and H.

Let $m \geq 1$ and choose functions $w_1, \ldots, w_m \in D(L^*) = H_0(J)$ to form an orthonormal set in S. Let S_0 be the m+n dimensional subspace of S spanned by $\phi_1, \phi_2, \ldots, \phi_n$ and $Hw_1, Hw_2, \ldots, Hw_m$. Clearly, $S_0 \in D(L)$. Let P_m and Q_m be projections defined by

$$Q_m x = \sum_{i=1}^{m} (x, w_i)w_i \quad \text{for all } x \in S.$$

and
$$P_m x = \sum_{i=1}^{n} (x, \phi_i)\phi_i + \sum_{i=1}^{m} (x, L^* w_i)Hw_i \text{ for all } x \in S.$$

Clearly P_m, Q_m are linear, continuous and idempotent ; and $R(Q_m) = \langle w_1, \ldots, w_m \rangle$ and $R(P_m) = S_0$. The operators P_m, Q_m, L and H satisfy the following lemma.

__Lemma 1.1.__ P_m, Q_m, L and H satisfy the following relations.

(i) $H(I-Q_m)Lx = (I-P_m)x$ for all $x \in D(L)$.

(ii) $LH(I-Q_m)x = (I-Q_m)x$ for all $x \in S$.

(iii) $LP_m x = Q_m Lx$ for all $x \in D(L)$.

(iv) $P_m H(I-Q_m)x = 0$ for all $x \in S$.

__Proof.__ (i) Let $x \in D(L)$. Then

$$(I-Q_m)Lx = Lx - \sum_{i=1}^{m} (Lx, w_i)w_i$$

$$= Lx - \sum_{i=1}^{m} (x, L^* w_i)w_i.$$

Therefore $H(I-Q_m)Lx = HLx - \sum_{i=1}^{m} (x, L^* w_i)Hw_i$

$$= x - \sum_{i=1}^{n} (x,\phi_i)\phi_i - \sum_{i=1}^{m} (x, L^* w_i)Hw_i.$$

$$= x - P_m x = (I - P_m)x.$$

(ii) Since $(I - Q_m)x \in R(L)$ for all $x \in S$, from (1.6) it follows that $LH(I - Q_m)x = (I - Q_m)x$.

(iii) Let $x \in D(L)$. Then

$$P_m x = \sum_{i=1}^{n} (x,\phi_i)\phi_i + \sum_{i=1}^{m} (x, L^* w_i)Hw_i$$

Therefore $LP_m x = \sum_{i=1}^{n} (x,\phi_i)L\phi_i + \sum_{i=1}^{m} (x, L^* w_i)LHw_i$

$$= \sum_{i=1}^{m} (x, L^* w_i)w_i \quad \text{(since } \phi_1, \ldots, \phi_n \in N(L) \text{ and } H \text{ is the right inverse of } L\text{)}$$

$$= \sum_{i=1}^{m} (Lx, w_i) w_i$$

$$= P_m Lx.$$

(iv) Let $x \in S$.

$$H(I-Q_m)x = H(x - \sum_{j=1}^{m} (x, w_j) w_j)$$

$$= Hx - \sum_{j=1}^{m} (x, w_j) Hw_j .$$

Hence $P_m H(I-Q_m)x = P_m Hx - P_m(\sum_{j=1}^{m} (x, w_j) Hw_j).$

$$= \sum_{i=1}^{n} (Hx, \phi_i)\phi_i + \sum_{i=1}^{m} (Hx, L^* w_i) Hw_i$$

$$- \sum_{i=1}^{n} (\sum_{j=1}^{m} (x, w_j) Hw_j, \phi_i)\phi_i$$

$$- \sum_{i=1}^{m} (\sum_{j=1}^{m} (x, w_j) Hw_j, L^* w_i) Hw_i .$$

$$= \sum_{i=1}^{m} (Hx, L^* w_i) Hw_i - \sum_{i=1}^{m} (\sum_{j=1}^{m} (x, w_j) w_j, w_i) Hw_i .$$

The last step follows from the fact that Hx and Hw_i, $i=1, \ldots m, \ldots$ belong to $N(L)^{\perp}$ and $\phi_1, \ldots, \phi_n \in N(L)$.

$$\text{Thus } P_m H(I-Q_m)x = \sum_{i=1}^{m} (LHx, w_i) Hw_i - \sum_{i=1}^{m} (x, w_i) Hw_i = 0.$$

We next give integral representations for H and $H(I-Q_m)$. Let $\Phi(t)$ be the fundamental matrix for τ formed by $\phi_1, \ldots, \phi_n$. Let $G(t,s)$ be the matrix function defined on $J \times J$ by

$$G(t,s) = \begin{cases} \Phi(t)\, \Phi^{-1}(s) & \text{for } a \leq s \leq t \leq b \\ 0 & \text{for } a \leq t \leq s \leq b \end{cases} \qquad (1.7)$$

We shall use the following result.

<u>Lemma 1.2.</u> Let $y \in S$ and let

$$u(t) = \int_a^t \Phi(t)\ \Phi^{-1}(s)y(s)ds,\ t \in J \qquad (1.8)$$

Then the function $u \in H(J)$ and $\tau u = y$.

<u>Proof</u>. $u^{(1)}(t) = \Phi(t)\ \Phi^{-1}(t)y(t) + \int_a^t \Phi^{(1)}(t)\ \Phi^{-1}(s)y(s)ds$

$$= y(t) + \int_a^t \Phi^{(1)}(t)\ \Phi^{-1}(s)y(s)ds.$$

Also $A(t)u = \int_a^t A(t)\ \Phi(t)\ \Phi^{-1}(s)y(s)ds.$

Hence $u^{(1)}(t) + A(t)u = y(t) + \int_a^t (\Phi^{(1)}(t)+A(t)\Phi(t))\,\Phi^{-1}(s)y(s)ds.$

$$= y(t) + \int_a^t \tau(\Phi(t))\,\Phi^{-1}(s)y(s)ds.$$

$= y(t)$ (since $\Phi(t)$ is a fundamental matrix for τ. See [17]).

We note that the above result shows also that $R(L) = S$. The following theorem gives a representation of H which is used subsequently.

<u>Theorem 1.1</u>. Let $y \in S$. Then Hy has a representation given by

$$(Hy)(t) = \sum_{i=1}^{n} \left(\int_a^b \Psi_i(s)y(s)ds\right)\phi_i(t) + \int_a^b G(t,s)y(s)ds,\ t \in J.$$

where the Ψ_is are given by

$$\Psi_i(s) = -\int_s^b \phi_i(t)\ \Phi(t)\ \Phi^{-1}(s)dt,\ i = 1,2,\ldots,n \quad (1.9)$$

<u>Proof</u>. Let $y \in S$. and $x = Hy$. Let $u(t)$ be given by (1.7). Using basic properties of H together with lemma 1.2, we get $x, u \in H(J)$ and $\tau(x-u) = 0$. Hence, $x = \sum_{i=1}^{n} c_i\phi_i + u$. Since $x \in N(L)$,$(x,\phi_i) = 0$ for $i = 1,2,\ldots,n$. But $(x,\phi_i) = c_i + (u,\phi_i)$.

Hence $c_i = -(u,\emptyset_i)$. Now $(u,\emptyset_i) = \int_a^b (\int_a^t G(t,s)y(s)ds)\emptyset_i(t)dt$

$$= \int_a^b\int_a^t (G(t,s)y(s)\emptyset_i(t))dsdt.$$

$$= \int_a^b\int_s^b (G(t,s)y(s)\emptyset_i(t))dtds \quad \text{by Fubini's theorem}$$

$$= \int_a^b (\int_s^b G(t,s)\emptyset_i(t)dt)y(s)ds.$$

$$= -\int_a^b \Psi_i(s)y(s)ds.$$

Hence $c_i = \int_a^b \Psi_i(s)y(s)ds$ and substituting in the expression for x we get the required representation.

We note that each Ψ_i is a row vector with n components. Let Φ_1 be the (nxn) matrix with Ψ_i occupying the i^{th} row. Let K(.,.) be the nxn matrix function defined on the square JxJ by

$$K(t,s) = \begin{cases} \Phi(t)(\Phi_1(s) + \Phi^{-1}(s)) & \text{for } a \leq s \leq t \leq b \\ \Phi(t)\ \Phi_1(s) & \text{for } a \leq t \leq s \leq b \end{cases} \tag{1.10}$$

We note that K(.,.) is continuous on JxJ except at the point t = s. We now state a corollary of Theorem 1.1.

<u>Corollary</u>: The right inverse operator H has an integral representation given by

$$(Hy)(t) = \int_a^b K(t,s)y(s)ds,\ t \in J \text{ for all } y \in S.$$

Let $K_m(.,.)$ be the matrix function defined on the square JxJ by

$$K_m(t,s) = K(t,s) - \sum_{i=1}^{m} (\int_a^b K(t,\xi)w_i(\xi)d\xi)w_i(s) \tag{1.11}$$

Let $K_m(t,s) = K^m_{ij}(t,s)$. We notice that $K^m_{ij}(t,s)$ are square integrable on JxJ, while the functions $\int_a^b K^m_{ij}(t,s)ds$ are

are continuous on $J(i=1,2,\ldots,n)$.

The following theorem gives an integral representation for $H(I-Q_m)$.

<u>Theorem 1.2</u>.

Let $y \in S$. Then $(H(I-Q_m)y)(t) = \int_a^b K_m(t,s)y(s)ds, t \in J$.

<u>Proof</u>. Since $(I-Q_m)x \in R(L)$, by the corollary of Theorem 1.1, we have

$$(H(I-Q_m)x)(t) = \int_a^b K(t,s)(I-Q_m)x(s)ds.$$

On the other hand,

$$\int_a^b K_m(t,s)x(s)ds = \int_a^b \Big[K(t,s) - \sum_{i=1}^m \Big(\int_a^b K(t,\xi)w_i(\xi)d\xi\Big)w_i(s)\Big]x(s)ds.$$

$$= \int_a^b K(t,s)x(s)ds.$$

$$- \sum_{i=1}^m \Big(\int_a^b K(t,\xi)w_i(\xi)d\xi\Big)\int_a^b w_i(s)x(s)ds.$$

$$= \int_a^b K(t,s)x(s)ds - \sum_{i=1}^m \int_a^b (x,w_i)K(t,s)w_i(s)ds.$$

$$= \int_a^b K(t,s)\Big[x(s) - \sum_{i=1}^m (x,w_i)w_i(s)\Big]ds.$$

$$= \int_a^b K(t,s)(I-Q_m)x(s)ds.$$

Thus it follows that $(H(I-Q_m)x)(t) = \int_a^b K_m(t,s)x(s)ds$. This completes the theorem.

We derive some estimates.

Let $x \in S$ and consider $\|H(I-Q_m)x\| = \|\int_a^b K_m(\cdot,s)x(s)ds\|$.

Obviously

$$\| H(I-Q_m)x \| = \Big(\sum_{i=1}^{n} \int_a^b \Big(\int_a^b \sum_{j=1}^{n} K_{ij}^m(t,s)x(s)ds\Big)^2 dt\Big)^{1/2}$$

$$\leq \Big(\sum_{i=1}^{n} \int_a^b \Big(\int_a^b \sum_{j=1}^{n} K_{ij}^m(t,s)^2 ds\Big) \|x\|^2 \, dt\Big)^{1/2}$$

(since $|(x,y)| \leq \|x\|\|y\|$)

$$= \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} \int_a^b \int_a^b K_{ij}^m(t,s)^2 ds dt\Big)^{1/2} \| x \|$$

Denoting by $\theta_m = \Big(\int_a^b \int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij}^m(t,s)^2 \, dsdt\Big)^{1/2}$, (1.12)

We have, $\| H(I-Q_m)x \| \leq \theta_m \| x \|$. (1.13)

Similarly we can show that $\mu(H(I-Q_m)x) \leq \hat{\theta}_m \| x \|$ (1.14)

where $\hat{\theta}_m = \max\limits_{i=1,2,\ldots,n} \Big(\sup\limits_{t \in J} \int_a^b \sum_{j=1}^{n} K_{ij}^m(t,s)^2 ds\Big)^{1/2}$ (1.15)

4. Boundary Operator M and some related properties.

Let $p_1 = \dim N(M)$ and $q_1 = \dim N(M^*)$. By assumption 2(iv) on M, we have $p_1 \geq q_1$. Choose functions $g_1, g_2, \ldots g_{p_1} \in D(M)$ to form an orthogonal basis for $N(M)$. As before, the restriction of M to the subspace $D(M) \cap N(M)^{\perp}$ is a 1-1 closed linear operator with closed range and its inverse $F = [M|_{D(M) \cap N(M)^{\perp}}]^{-1}$ is a 1-1 continuous linear operator with domain $R(M)$, range $D(M) \cap N(M)^{\perp}$, and

$$MFg = g \text{ for all } g \in R(M).$$

$$FMg = g - \sum_{i=1}^{p_1} (g, g_i)g_i \text{ for } g \in D(M) \qquad (1.16)$$

We observe that F is a continuous right inverse of M.

We now define projections P'_{m_1} and Q'_{m_1} in S_1 and study their relations with M and F. Choose functions $h_1, h_2, \ldots, h_{q_1}$ in S_1 to form an orthonormal basis for $N(M^*)$. Let $m_1 \geq q_1$ be any integer and choose functions $h_{q_1+1}, \ldots, h_{m_1}$ in $D(M^*)$ such that the functions $h_1, \ldots, h_{q_1}, h_{q_1+1}, \ldots, h_{m_1}$ form an orthonormal set in S_1. We observe that $S_1 = R(M) \oplus N(M^*)$, where $\oplus$ denotes the orthogonal direct sum . Note that the functions $h_{q_1+1}, \ldots, h_{m_1}$, belong to $R(M)$ and hence, we can form the functions $Fh_{q_1+1}, \ldots, Fh_{m_1}$. Let S'_o be the $(p_1+m_1-q_1)$ ϵ dimensional subspace of S_1 spanned $g_1, \ldots, g_{p_1}$ and $Fh_{q_1+1}, \ldots, Fh_{m_1}$. Clearly $S'_o \in D(M)$.

Define the projections P'_{m_1} and Q'_{m_1} in S_1 by

$$P'_{m_1} g = \sum_{i=1}^{p_1} (g, g_i)_1 g_i + \sum_{i=q_1+1}^{m_1} (g, M^* h_i)_1 Fh_i \text{ for all } g \in S_1.$$

$$Q'_{m_1} g = \sum_{i=1}^{m_1} (g, h_i)_1 h_i \text{ for all } g \in S_1.$$

Clearly P'_{m_1} and Q'_{m_1} are continuous, linear and idempotent operators defined on all of S_1, and $R(P'_{m_1}) = S'_o$ and $R(Q'_{m_1}) = \langle h_1, \ldots, h_{m_1} \rangle$. Also, the range of $(I_1 - Q_{m_1})$ is a subset of $R(M)$ and $F(I'_1 - Q'_{m_1})$ is a continuous linear operator defined on all of S_1. As before, operators P'_m, Q'_m, F and M satisfy the following properties.

<u>Lemma 1.3</u> : **The following properties are valid.**

(i) $F(I_1-Q'_{m_1})Mg = (I_1-P'_{m_1})g$ for all $g \in D(M)$

(ii) $MF(I_1-Q'_{m_1})g = (I_1-Q'_{m_1})g$ for all $g \in S_1$.

(iii) $MP'_{m_1}g \quad = \quad Q'_m Mg$ for all $g \in D(M)$.

(iv) $P'_{m_1}F(I_1-Q'_{m_1}) = 0$ for all $g \in S_1$.

Finally, let θ'_{m_1} and $\hat{\theta}'_{m_1}$ be the numbers such that

$$\| F(I_1 - Q'_{m_1})g \| \le \theta'_{m_1} \| g \|_1 \text{ for all } g \in S_1 \tag{1.17}$$

$$\mu_1(F(I_1-Q'_{m_1})g) \le \hat{\theta}'_{m_1} \| g \|_1 \text{ for all } g \in \tilde{S}_1 \tag{1.18}$$

whenever the L.H.S. of (1.16) is finite.

It will be assumed that F maps essentially bounded functions into essentially bounded functions.

5. Existential Analysis :

Let $Y = S \times S_1$, the product space of S and S_1 with norm $\| (x,g) \|_Y = \| x \| + \| g \|_1$, $x \in S$, $g \in Y_1$.

Choose functions $x_o \in S_o$ and $g_o \in S_1$ such that $\beta = \mu(x_o) < R$ and $\beta_1 = \mu_1(g_o) < R$. Define,

$$\hat{x}_o(t) = \begin{cases} g_o(t) + (F(I_1-Q'_{m_1})B(x_o)_b)(t) & \text{for } t \in [a-\nu, a] \\ x_o(t) & \text{for } t \in [a,b] \end{cases}$$

Obviously, $\hat{x}_o$ is an extension of x_o to the whole of $[a-\nu, b]$.

Let $\mu_1(F(I_1-Q'_{m_1})B(x_o)_b) = \hat{e}_1$.

Therefore, $\mu_1(g_o+F(I_1-Q'_{m_1})B(x_o)_b) \le \beta_1+\hat{e}_1$. It will be assumed that

$$\beta_1+\hat{e}_1 < R. \tag{1.19}$$

Let $Z_o = H(I-Q_m)G_{g_o}(x_o)$ where $(G_{g_o}(x_o))(t) = X(t,(x_o)_t), t \in J$. Suppose e and $\hat{e}$ are real constants satisfying $\|Z_o\| \leq e$, $\mu(Z_o) \leq \hat{e}$. Denoting by $g_o = F(I_1-Q'_{m_1})B(x_o)_b$, we have $\|g_o\|_1 \leq e_1$ and $\mu_1(g_o) \leq \hat{e}_1$. Assume that c, d, r and R are positive real numbers satisfying

$$c+e < d,\ r+\hat{e} < \hat{R},\ \hat{R}+\beta \leq R \text{ and } r+\beta_1+\hat{e}_1 < R \tag{1.20}$$

The sets V in Y and $\tilde{S}_o$ in S are defined as follows :

$$V = \{ (x,g) \in S_o x S_o : \|x-x_o\| + \|g-g_o\|_1 \leq c, \\ \max (\mu(x-x_o), \mu_1(g-g_o)) \leq r \}. \tag{1.21}$$

and $\tilde{S}_o = \{ x \in \tilde{S} \mid \|x-x_o\| \leq d,\ \mu(x-x_o) \leq \hat{R} \}$. (1.22)

Clearly V and $\tilde{S}_o$ are nonempty closed, bounded subsets in their respective spaces and $(x,g) \in V$ implies $x \in \tilde{S}_o$.

Let $(x^*, g^*) \in V$ and $x \in \tilde{S}_o$. Define $x(g^*)$ by

$$x(g^*) = \begin{cases} g^* + F(I_1-Q'_{m_1})Bx_b & \text{on } [a-\gamma, a] \\ x & \text{on } [a,b] \end{cases}$$

We note that $\mu_1(g^* + F(J_1-P'_{m_1})Bx_b) \leq R$ for x_b essentially bounded and sufficiently large m_1. Indeed,

$$\mu_1(g + F(I_1-Q'_{m_1})Bx_b) \leq \mu_1(g-g_o) + \mu_1(g_o) + \mu_1(F(I_1-Q'_{m_1})(Bx_b-B(x_o)_b) \\ + \mu_1(F(I_1-Q'_{m_1})B(x_o)_b)$$

$$\leq r+\beta_1+\hat{\theta}'_{m_1} \|Bx_b-B(x_o)_b\| + \hat{e}_1 \text{(by (1.21) and (1.18))}$$

$$\leq r+\beta_1+\hat{e}_1 + \hat{\theta}'_{m_1} kd \text{ (by (1.4) and (1.22))}.$$

If m_1 is taken large enough we have

$$\theta'_{m_1} kd \leq R-(r+\beta_1+e_1) \tag{1.23}$$

For each $(x^*, g^*) \in V$ and $x \in \tilde{S}_o$, define $(G_{g^*}(x))(t) = X(t, x(g^*)_t)$, $t \in J$. Obviously, $G_{g^*}(x)$ is well defined on $\tilde{S}_o$. Further,

$$||G_{g^*}(x)-G_{g^{**}}(y)|| = \left(\sum_{i=1}^{n} \int_a^b |X_i(t,x(g^*)_t)-X_i(t,y(g^{**})_t)|^2 dt\right)^{1/2}$$

$$\leq \left(\sum_{i=1}^{n} k_i^2 \int_a^b \sum_{j=1}^{p} \sum_{\ell=1}^{n} x_\ell(g^*)(t+\theta_j)\right.$$

$$\left. -y_\ell(g^{**})(t+\theta_j))^2 dt\right)^{1/2} \text{ by } (1.3)$$

$$\leq \left(\sum_{i=1}^{n} k_i^2\right)^{1/2} \sum_{j=1}^{p} \sum_{\ell=1}^{n} \int_{a+\theta_j}^{a} (x_\ell(g^*)(t)-y_\ell(g^{**})(t))^2 dt$$

$$+ \int_a^{b+\theta_j} (x_\ell(g^*)(t)-y_\ell(g^{**})(t))^2 dt \,]^{1/2}.$$

$$\leq \left(p \sum_{i=1}^{n} k_i^2\right)^{1/2} \sum_{\ell=1}^{n} \int_{a-\nu}^{a} g_\ell^*(t)+(F(I_1-Q'_{m_1})\ (t)-g^{**}(t)$$

$$-(F(I_1-Q'_{m_1})By_b)\ (t)\,]^2\ dt$$

$$+ \int_a^b (x\ (t)\ -y\ (t))^2\ dt\,]^{1/2}$$

$$= \left(p \sum_{i=1}^{n} k_i^2\right)^{1/2} [\,||g^*-g^{**} + F(I_1-Q'_{m_1})(Bx_b-By_b)||_1^2+||x-y||^2]^{1}$$

$$\leq \left(p \sum_{i=1}^{n} k_i^2\right)^{1/2} [\,(||g^*-g^{**}||_1+\theta'_{m_1} k||x_b-y_b||_1)^2+||x-y||^2]^{1/2}$$

(by the triangle inequality and relations (1.17) and (1.4)).

$$\leq \left(p \sum_{i=1}^{n} k_1^2\right)^{1/2} [\,||g^*-g^{**}||_1+(1+\theta'_{m_1} k)||x-y||\,]$$

Denote by $k_o = \left(p \sum_{i=1}^{n} k_i^2\right)^{1/2}$ and by $k_o = k_o(1+\theta'_{m_1} k)$.

Then, we get,

$$||G_{g^*}(x)-G_{g^{**}}(y)|| \leq k_o(||g^*-g^{**}||_1+(1+\theta'_{m_1}k)||x-y||) \quad (1.24)$$

In case $g^* = g^{**} = g$, we have $||G_g(x)-G_g(y)|| \leq k_o||x-y|| \quad (1.25)$

For each point $(x^*, ^*) \in V$, define the map T on $\tilde{S}_o$ as follows:

$$Tx = x^* + H(I-Q_m)G_{g^*}(x), \; x \in \tilde{S}_o.$$

suppose T has a fixed point. Then there exists an $x \in \tilde{S}_o$ such that $x = x^* + H(I-Q_m)G_{g^*}(x)$. Operating by L on both sides and using lemma 1.1, we get $Lx-G_{g^*}(x) = Q_m(Lx-G_{g^*}(x))$. Also by the definition of $x(g^*)$, we note that $x_a = g^* + F(I_1-Q'_{m_1})Bx_b$. Operating by M on both sides and using lemma 1.3, we get $Mx_a-Bx_b = Q'_{m_1}(Mx_a-Bx_b)$. Therefore $x(g^*,x^*)$ is a solution of the given boundary value problem (1.1)-(1.2) provided $Q_m(Lx-G_{g^*}(x)) = 0$ and $Q'_{m_1}(Mx_a-Bx_b) = 0$. Last two equations are called the bifurcation equations. We use the contraction mapping principle to get a fixed point of T and the degree theory to solve the bifurcation equations. The following theorem gives the continuous fixed point of T.

<u>Theorem 1.3:</u>

Let the assumptions of section 2 and conditions (1.17),(1.18), (1.20),(1.23) be satisfied. Let 'm' and 'm_1' be sufficiently large such that

$$\begin{aligned} &c+e+\theta_m k_o(c+(1+\theta'_{m_1}k)d) \leq d, \\ &r+\hat{e}+\theta_m k_o(c+1+\theta'_{m_1}k)d) \leq \hat{R}, \text{ and} \qquad (1.26) \\ &\theta_m k_o(1+\theta'_{m_1}k) < 1 \end{aligned}$$

Then T has a unique fixed point in $\tilde{S}_o$ and the fixed point varies continuously with respect to (x^*, g^*).

Proof.: Firstly, we show that $T(\tilde{S}_o) \subseteq \tilde{S}_o$. For, let $x \varepsilon S_o$. Then

$$||Tx-x_o|| = ||x^*-x_o+H(I-Q_m)G_g*(x)||$$

$$\leq ||x-x_o||+||H(I-Q_m)(G_g*(x)-G_g(x_o))||+||H(I-Q_m)G_{g_o}(x_o)||$$

$$\leq c+\theta_m||G_g*(x)-G_{g_o}(x_o)||+e \text{ by (1.13) and (1.21)}$$

$$\leq c+e+\theta_m k_o(||g^*-g_o||_1+(1+\theta'_{m_1}k)||x-x_o|| \quad \text{by (1.24)}$$

$$\leq c+e+\theta_m k_o \ (c+(1+\theta'_{m_1}k)d) \quad \text{by (1.22)}$$

$$\leq d \quad \text{by (1.26)}$$

Similarly, we can show that $\mu(Tx-x_o) \leq \hat{R}$. Thus $T(\tilde{S}_o) \subseteq \tilde{S}_o$. We now prove that T is a contraction. For. let $x,y \varepsilon \tilde{S}_o$. Then

$$||Tx-Ty|| = ||H(I-Q_m)(G_g*(x)-G_g*(y))||$$

$$\leq\theta_m|| G_g*(x)-G_g*(y)|| \quad \text{by (1.13)}$$

$$\leq\theta_m k_o(1+\theta'_{m_1}k)||x-y|| \quad \text{by (1.25)}$$

Now, relation (1.26) shows that T is a contraction.

Hence by the contraction mapping principle T has unique fixed point in S_o. Finally, we show the continuous dependence. For, let $x(x^*, g^*)$ and $y(y^*, g^{**})$ be the fixed points of T corresponding to (x^*, g^*), and $(y^*; g^{**})$, respectively.

Then

$$||x(x^*,g^*)-y(y^*,g^{**})|| = ||x^*-y^*+H(I-Q_m)(G_g*(x)-G_g**(y))||$$

$$\leq ||x^*-y^*||+\theta_m k_o(||g^*-g^{**}||_1)+(1+\theta'_{m_1}k)||x-y|| \text{ by (1.13)}$$

and (1.24).

$$\leq \|x^*-y^*\| + \|g^*-g^{**}\|_1 + \theta_m k_o(1+\theta'_{m_1} k)\|x-y\| \quad (\text{since } \theta_m k_o<1).$$

Hence,

$$\|x(x^*,g^*)-y(y^*,g^{**})\| \leq (1-\theta_m k_o(1+\theta'_{m_1} k))^{-1}(\|y^*-x^*\|+\|g^*-g^{**}\|_1).$$

In view of the fact that $\theta_m k_o(1+\theta'_{m_1} k) < 1$, the continuity follows and the proof is completed.

We now assert that the extension of $x(x^*,g^*)$ to the whole interval $[a-\nu, b]$ varies continuously with respect to (x^*,g^*). For, consider

$$x(x^*,g^*)_a = g^*+F(I_1-Q'_{m_1})Bx_b \text{ and}$$

$$y(y^*,g^{**})_a = g^{**}+F(I_1-Q'_{m_1})By_b.$$

Then $\|x(x^*,g^*)_a-y(x^*,g^*)_a\|_1 = \|g^*-g^{**}\|_1+\|F(I_1-Q'_{m_1})(Bx_b-By_b)\|_1$

$$\leq \|g^*-g^{**}\|_1+\theta'_{m_1} k\|x_b-y_b\|_1 \text{ by (1.17) and (1.4).}$$

$$\leq \|g^*-g^{**}\|_1+\theta'_{m_1} k\|x-y\|$$

From the above inequality and the continuous dependence of x on (x^*,g^*), the assertion follows readily.

Bounds for the solution :

Since $x \in S_o$, we have $\|x-x_o\| \leq d$ and $\mu(x-x_o) \leq R$. Also, as in (1.27), we get $\|x_a-(x_o)_a\|_1 \leq c+\theta'_{m_1} kd$. Similarly, we can get $\mu_1(x_a-(x_o)_a) \leq r+\theta'_{m_1} kd$.

Now we shall present some theorems for solving the bifurcation equations. For that we need the following.

Let $(x^*,g^*) \in V$ and $\Gamma(x^*,g^*)$ be the fixed point of T

corresponding to (x^*,g^*). Clearly, $\Gamma: V \to \tilde{S}_o$ is continuous. Define by

$$\Psi(x^*,g^*) = (Q_m(Lx - G_{g^*}(x^*)),\ Q'_{m_1}(Mx\ (g^*)-Bx^*_b))$$

and $\Psi\Gamma(x^*,g^*) =$

$$=Q_m(L\ (x^*,g^*)-G_{g^*}(\Gamma(x^*,g^*))),Q'_{m_1}(M\hat{\Gamma}(x^*,g^*)_a-B\Gamma(x^*,g^*)_b)).$$

Recall that $\hat{\Gamma}$ is an extension of Γ.

We assert that $\Psi\Gamma: V \to \langle w_1,\ldots,w_m\rangle \times \langle h_1,\ldots,h_{m_1}\rangle$ is continuous. For, consider

$$\begin{aligned}||\Psi\Gamma(x^*,g^*)-\Psi\Gamma(y^*,g^{**})||_Y &= ||Q_mL\Gamma(x^*,g^*)-L\Gamma(y^*,g^{**})\\
&\quad + G_{g^{**}}(\Gamma(y^*,g^*)-G_{g^*}(\Gamma(x^*,g^*)))||\\
&\quad + ||Q'_{m_1}(M\hat{\Gamma}(x^*,g^*)_a-M\hat{\Gamma}(y^*,g^{**})_a\\
&\quad + B\Gamma(y^*,g^{**})_b-N\Gamma(x^*,g^*)_b||_1.\\
&\le ||\sum_{i=1}^{m}(\Gamma(x^*,g^*)-(y^*,g^{**}),\ L^* w_i)w_i|| +\\
&\quad + ||G_g\ (\Gamma(x^*,g^*))-G_{g^{**}}(\Gamma(y^*,g^{**}))||\\
&\quad + ||\sum_{i=1}^{m_1}(\hat{\Gamma}(x^*,g^*)_a-\hat{\Gamma}(y^*,g^{**})_a,M^*h_i)h_i||_1\\
&\quad + ||N\Gamma(x^*,g^*)_b-B\ (y^*,g^{**})_b||_1\\
&\le \sum_{i=1}^{m}||L^*w_i||\ ||\ (x^*,g^*)-(y^*,g^{**})||\\
&\quad + \sum_{i=1}^{m_1}||M^*h_i||_1\ ||\hat{\Gamma}(x^*,g^*)_a-\hat{\Gamma}(y^*,g^{**})_a||_1\\
&\quad + k_o(||g^*-g^{**}||_1+(1+\theta'_{m_1}k)||\Gamma(x^*,g^*)-\Gamma(y^*,g^{**})||\ )\\
&\quad + k\ ||\Gamma(x^*,g^*)_b-\Gamma(y^*,g^{**})_b||_1 \text{ by (1.24) and (1.4).}\end{aligned}$$

above inequality shows that $\Psi\Gamma$ is continuous. Similarly,

we can show that $\Psi : V \to \langle w_1,\dots,w_m\rangle \times \langle h_1,\dots,h_{m_1}\rangle$ is continuous.

We need the following estimate.

<u>Lemma 1.4</u>. Let $(x^*,g^*) \in V$. Then

$$||\Psi\Gamma(x^*,g^*) - \Psi(x^*,g^*)||_Y \leq (k+k_o(1+\theta'_{m_1}k)) \quad \theta_m k_o(c+(1+\theta'_{m_1}k)d)+e \quad .$$

<u>Proof</u>.

Clearly, $Q_m L\Gamma(x^*,g^*) = Q_m Lx^*$ and $Q'_{m_1} M\hat{\Gamma}(x^*,g^*)_a = Q'_{m_1} Mx^*(g^*)_a = Q'_{m_1} Mg^*$. This follows from the definition of T and extension of a function when one makes use of lemmas 1.1 and 1.3. Therefore,

$$||\Psi\Gamma(x^*,g^*) - \Psi(x^*,g^*||_Y$$

$$= ||Q_m(G_{g^*}(\Gamma(x^*,g^*))-G_{g^*}(x^*)|| + ||Q'_{m_1}(B\Gamma(x^*,g^*)_b - Bx^*_b)||_1$$

$$\leq k_o(1+\theta'_{m_1}k)||\Gamma(x^*,g^*)-x|| + k||\Gamma(x^*,g^*)_b - x^*_b||_1 \text{ by}$$

(1.25) and (1.4)

$$\leq (k+k_o(1+\theta'_{m_1}k))||\Gamma(x^*,g^* - x^*||$$

$$\leq (k+k_o(1+\theta'_{m_1}k))(||H(I-Q_m)(G_g(\Gamma(x^*,g^*))-G_{g_o^*}(x_o))||$$

$$+ ||H(I-Q_m)G_{g_o}(x_o)||).$$

$$\leq (k+k_o(1+\theta'_{m_1}k))\ \theta_m k_o(||g^*-g_o||_1 + (1+\theta'_{m_1}k)||\Gamma(x^*,g^*)-x_o||)+e$$

by (1.13) and (1.24)]

$$\leq (k+k_o(1+\theta'_{m_1}k))[\theta_m k_o(c+1+\theta'_{m_1}k)d+e]$$

The proof is completed.

Apply the Gram-Schmidt process to the functions $Hw_1,\dots,Hw_m$

to obtain the orthonormal functions $\eta_1, \eta_2, \ldots, \eta_m$ and to the functions $Fh_{q_1+1}, \ldots, Fh_{m_1}$ to obtain the orthonormal functions $\eta'_1, \eta'_2, \ldots, \eta'_{m_1}$. Let $M = m+n+p_1+m_1-q_1$. Let E^M be a copy of Euclidean M-space where we represent each point ξ by an M-tuple of numbers.

$$\xi = (b_1, b_2, \ldots, b_n, c_1, \ldots, c_m, b'_1, \ldots, b'_{p_1}, c'_{q_1+1}, \ldots, c'_{m_1}).$$

Let $\hat{m} = m+m_1$. Let $E^{\hat{m}}$ be a copy of Euclidean $\hat{m}$-space where we represent each point $V \varepsilon E^{\hat{m}}$ as an $\hat{m}$-tuple : $V=(u_1, \ldots u_m, u'_1, \ldots, u'_{m_1})$. Define the operators $\Gamma_1 : E^M \to \delta_0 x\, \delta'_0$ and $\Gamma_2 : \langle w_1, \ldots, w_m \rangle x \langle h_1, \ldots, h_{m_1} \rangle \to E^{\hat{m}}$ by

$$\Gamma_1(b_1, b_2, \ldots, b_n, c_1, \ldots, c_m; b'_1, \ldots, b'_{p_1}, c'_{q_1+1}, \ldots, c'_{m_1})$$

$$= \left(\sum_{i=1}^{n} b_i \emptyset_i + \sum_{i=1}^{m} c_i \eta_i, \quad \sum_{i=1}^{p_1} b'_i g_i + \sum_{i=q_1+1}^{m_1} c'_i \eta'_i \right).$$

and $\Gamma_2\left(\sum_{i=1}^{m} u_i w_i, \quad \sum_{i=1}^{m_1} u'_i h_i \right) = (u_1, \ldots, u_m, u'_1, \ldots, u'_{m_1})$.

It is not difficult to check that Γ_1 and Γ_2 are isomorphisms.

Let $\Gamma_1(\xi_0) = (x_0, g_0)$. Let $\Psi : E^M \to E^{\hat{m}}$ be the operator defined by

$$\Psi = \Gamma_2 \Psi \Gamma_1.$$

Let us choose a number $\varepsilon > 0$ such that the set U = $U = \{ \xi \varepsilon E^M | \ |\xi - S_0| \leq \varepsilon \}$ is mapped by Γ_1 into the set V. The existence of such an ε is not difficult to show. We observe that $\Gamma_2 \ \Gamma_1$ and $\Gamma_2 \ \Gamma \Gamma_1$ map the ball $U \subseteq E^M$ continuously into

$\hat{E}^m$. This is used in the following theorems to establish the existence of a solution to the equation $\Psi\Gamma(x^*, g^*) = 0$ and hence a solution of the problem (1.1)-(1.2).

The following theorem establishes the existence of a solution for the case m = 1 and $m_1 = 0$. For $m_1 = 0$, we mean the operator M is invertible and the projections Q'_{m_1} are not considered. In this case, we denote θ'_{m_1} by θ' and $\hat{\theta}'_{m_1}$ by $\hat{\theta}'$.

<u>Theorem 1.4</u> : Let m = 1, $m_1 = 0$ and conditions of Theorem 1.3 be satisfied. Suppose there exists a number $\delta > 0$ such that $[-\delta, \delta] \subseteq \bar{\Psi}(U)$ and

$$(k+k_o(1+\theta' k))(\theta_m k_o(c+(1+\theta' k)d)+e) \leq \delta .$$

Then there exists an $(x^*, g^*) \in V$ such that $\Psi\Gamma(x^*, g^*) = 0$. Moreover, $L(x^*, g^*) = G_{g^*}(\Gamma(x^*, g^*))$ and $\hat{M}\Gamma(x^*, g^*) = B\Gamma(x^*, g^*)_b$, and $\hat{\Gamma}(x^*, g^*)$ is a solution of (1.1)-(1.2).

For the method of proof see Theorem 4 of [45].

The next theorem relaxes the conditions $m_1 = 0$ and m = 1.

<u>Theorem 1.5</u> : Let ξ_o be such that $\bar{\Psi}(\xi_o) = 0$ and let the conditions of Theorem 1.3 be satisfied. Let the first order partial derivatives of $\bar{\Psi}$ exist and are continuous on U and the Jacobian matrix for $\bar{\Psi}$ has rank $\hat{m}$ at ξ_o. Let the set $W = \{ u \in \hat{E}^m \mid |u| \leq \delta \}$ be a subset of $\bar{\Psi}(U)$ and there exists a continuous mapping $\Lambda : W \to U$ such that $\bar{\Psi}\Lambda(u) = u$ for all $u \in W$ and

$$(k+k_o(1+\theta'_{m_1} k))(\theta_m k_o(c+1+\theta'_{m_1} k)d+e) < \delta \quad \text{with } \delta > 0.$$

Then there exists an $(x^*, g^*) \in V$ such that $\Gamma(x^*, g^*) = 0$ and $\Psi \hat{\Gamma}(x^*, g^*)$ is a solution of (1.1)-(1.2).

For the method of proof see Theorem 5 of [45].

6. Periodic Boundary Conditions :

We now show that in the case of periodic boundary conditions (that is $M = B = I_1$) most of our estimates have simpler form. In this case we note that there is no need to consider the projections Q'_{m_1} in as much as $F = I_1$ is defined on all of S_1. Further the extension $\hat{x}$ of $x \in S$ is simply by $\hat{x}(t) = x(t)$ for $t \in [a,b]$ and $= x(t+b-a)$ for $t \in [a-\nu, a]$. One can easily verify that condition (1.20) takes the form

$$c+e<d, \quad r+\hat{e}<\hat{R} \text{ and } \hat{R}+\beta\leq R \tag{1.28}$$

and, (1.24) and (1.25) take the form

$$||G(x)-G(y)|| \leq k_o||x-y|| \tag{1.29}$$

We also note that S'_o is a singleton consisting of the zero element. Also the sets V and $\tilde{S}_o$ are simply defined by

$$V = \{ x \in S_o \mid ||x-x_o|| \leq C, \ \mu(x-x_o) \leq r \} .$$

$$\tilde{S}_o = \{ x \in \tilde{S} \mid ||x-x_o|| \leq d, \ \mu(x-x_o) \leq \acute{R} \}.$$

Thus the set V is a subset of the space S.

Thus in this case Theorem 1.3 takes the following simpler form.

<u>Theorem 1.6</u> : Suppose the assumptions 2(i)-2(iii) and condition (1.28) are satisfied. Let 'm' be sufficiently large such that $c+e+\theta_m k_o d \leq d$, $r+\hat{e}+\hat{\theta}_m k_o d \leq R$ and $\theta_m k_o < 1$. Then T has a unique fixed point in S_o and the fixed point varies continuously with respect to x^*.

In this case $M = n+m$, $\hat{m} = m$ and the map Ψ takes the form $\Psi(x) = Q_m(Lx-G(x))$ and lemma 1.4 becomes

<u>Lemma 1.5</u> : Let $x^* \in V$. Then $\| \Psi \tilde{\Gamma} x^* - \Psi x^* \| \leq (\theta_m k_o d+e) k_o$.

Here, the isomorphisms are $\Gamma_1 : E^M \to S_o$ and $\Gamma_2 : \langle w_1, \ldots, w_m \rangle \to E^m$. Finally, Theorems (1.7) and (1.8) correspond to the earlier Theorems (1.4) and (1.5) respectively.

<u>Theorem 1.7</u> : Let $m = 1$ and conditions of Theorem 1.6 be satisfied. Let $\delta > 0$ be a number such that $[-\delta, \delta] \subseteq \bar{\Psi}(U)$ and $k_o(\theta_m k_o d+e) \leq \delta$. Then there exists an $x \in \tilde{S}_o$ such that $\hat{x}$ satisfies (1.1) and $x_a = x_b$. Moreover, $\|x-x_o\| \leq d$ and $\mu(x-x_o) \leq \hat{R}$.

<u>Theorem 1.8</u> : Let ξ_o be such that $\bar{\Psi}(\xi_o) = 0$ and let the conditions of Theorem 1.6 be satisfied. Suppose the first order partial derivatives of $\bar{\Psi}$ exist and are continuous on U and the Jacobian matrix for $\bar{\Psi}$ has rank m at ξ_o. Let the set $W = \{ u \in E^m \mid |u| \leq \delta \}$ be a subset of $\bar{\Psi}(U)$ and there exists a continuous map $\Lambda : W \to U$ such that $\bar{\Psi}\Lambda(u) = u$ for all $u \in W$ and $(\theta_m k_o d+e)\, k_o < \delta$ with $\delta > 0$. Then there exists an $x \in \tilde{S}_o$ such that $\hat{x}$ satisfies (1.1) and $x_a = x_b$.

Remark : Our method gives solutions which may be discontinuous at the initial point $t = a$. In the simple case of periodic boundary conditions, the existence of solutions with no discontinuity at $t = a$ were established in [23].

CHAPTER-2

We shall make use of the theory developed in Chapter One and prove the existence of a solution to the following Van der Pol functional differential equation with periodic boundary conditions.

$$y^{(2)} + \alpha(1-y^2)y^{(1)} = \beta(y(t-1/4) + \sin 2\pi t), \quad |\alpha| \leq 0.1, |\beta| \leq 0.1$$

$$y_o = y_1 \text{ and } y_o^{(1)} = y_1^{(1)} \tag{2.1}$$

over the interval $J = [0,1]$. Here $J_1 = [-1/4, 0]$.

Equation of the above type and similar ones occur widely in applications and describe many interesting types of phenomena. They arise in the application of probabilistic methods to the theory of asymptotic prime number density, in the mathematical description of a fluctuating population of organisms under certain environmental conditions, in the operation of a control system working with potentially explosive chemical reactions and in the economic studies of business cycles. See G.S. Jones [39]. Writing equation (2.1) in system form, we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{(1)} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha(1-x_1^2)x_2+\beta(x_1(t-1/4)+\sin 2\pi t) \end{pmatrix} \tag{2.2}$$

where $x_1 = y$ and $x_2 = y^{(1)}$. Denoting $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have $\tau x = x^{(1)} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$. Therefore, the operator L is defined by

$D(L) = H(J)$, $Lx = \tau x$. We can easily check that the adjoint L^* of L is given by $D(L^*) = \{ z \in H(J) : z(0) = z(1) = 0 \}$, $L^*z = z^{(1)} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z$. Clearly, $\dim N(L) = 2$ and $\dim N(L^*) = 0$, and the functions $\phi_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\phi_2(t) = (\frac{12}{13})^{1/2} \begin{pmatrix} t-1/2 \\ 1 \end{pmatrix}$ form an orthonormal basis for $N(\tau)$. Simple calculation gives

$$\Phi(t) = \begin{pmatrix} 1 & (\frac{12}{13})^{1/2}\,(t-1/2) \\ 0 & (\frac{12}{13})^{1/2} \end{pmatrix}, \quad \phi^{-1}(s) = \begin{pmatrix} 1 & \frac{1}{2} - s \\ 0 & (\frac{13}{12})^{1/2} \end{pmatrix}$$

$$\text{and } \Phi_1(s) = \begin{pmatrix} s-1 & s - \frac{s^2}{2} - \frac{1}{2} \\ (\frac{12}{13})^{1/2}\,(\frac{s^2-s}{2}) & \frac{3s^2-2s^3+12s}{(156)^{1/2}} - (\frac{13}{12})^{1/2} \end{pmatrix}$$

Hence we get

$$K(t,s) = \begin{cases} \begin{pmatrix} 1 & (\frac{12}{13})^{1/2}\,(t - \frac{1}{2}) \\ 0 & (\frac{12}{13})^{1/2} \end{pmatrix} \begin{pmatrix} s & -\frac{s^2}{2} \\ (\frac{12}{13})^{1/2}(\frac{s^2-s}{2}) & \frac{3s^2-2s^3+12s}{(156)^{1/2}} \end{pmatrix} & 0 \le s < t \le 1, \\[2ex] \begin{pmatrix} 1 & (\frac{12}{13})^{1/2}(t - \frac{1}{2}) \\ 0 & (\frac{12}{13})^{1/2} \end{pmatrix} \begin{pmatrix} s-1 & s - \frac{s^2}{2} - 1 \\ (\frac{12}{13})^{1/2}(\frac{s^2-s}{2}) & \frac{3s^2-2s^3+12s}{(156)^{1/2}} - (\frac{13}{12})^{1/2} \end{pmatrix} & 0 \le t < s \le 1. \end{cases}$$

We can verify that $w_1(t) = (\frac{15}{8})^{1/2}\begin{pmatrix}\sin 2\pi t\\ t^2-t\end{pmatrix}$ is a normalized vector belonging to $D(L^*)$ and

$$\int_0^1 K(t,s)w_1(s)\,ds = (\tfrac{15}{8})^{1/2}\begin{pmatrix}\frac{t^4}{12}-\frac{t^3}{6}+\frac{t}{12}-\frac{1}{60}-\frac{\cos 2\pi t}{2\pi}\\[2ex] \frac{t^3}{3}-\frac{t^2}{2}+\frac{1}{12}\end{pmatrix}$$

Letting

$$\frac{15}{8}\begin{pmatrix}\frac{t^4}{12}-\frac{t^3}{6}+\frac{t}{12}-\frac{1}{60}-\frac{\cos 2\pi t}{2\pi}\\[2ex] \frac{t^3}{3}-\frac{t^2}{2}+\frac{1}{12}\end{pmatrix} = \begin{pmatrix}\alpha_1\\[2ex] \alpha_2\end{pmatrix},$$ one easily writes

$$K_1(t,s)=\begin{cases}\begin{pmatrix}s+\frac{6}{13}(t-\frac{1}{2})(s^2-s) & -\frac{s^2}{2}+\frac{1}{13}(t-\frac{1}{2})(3s^2-2s^3+12s)\\ -\alpha_1 \sin 2\pi s & -\alpha_1(s^2-s)\\[1ex] \frac{6}{13}(s^2-s)-\alpha_2 \sin 2\pi s & \frac{1}{13}(3s^2-2s^3+12s)-\alpha_2(s^2-s)\end{pmatrix} & \text{for } 0\le s\le t\le 1,\\[3ex] \begin{pmatrix}s-1+\frac{6}{13}(t-\frac{1}{2})(s^2-s) & s-\frac{s^2}{2}+\frac{1}{13}(t-\frac{1}{2})(3s^2-2s^3+12s)\\ -\alpha_1 \sin 2\pi s & -t-\alpha_1(s^2-s)\\[1ex] \frac{6}{13}(s^2-s)-\alpha_2 \sin 2\pi s & \frac{1}{13}(3s^2-2s^3+12s)-1-\alpha_2(s^2-s)\end{pmatrix} & \text{for } 0\le t< s\le 1\end{cases}$$

Lengthy calculations yield

$$\int_0^1 K_{11}^1(t,s)^2 ds = \frac{1}{3}+\frac{12}{1690}(t-\tfrac{1}{2})^2+\frac{\alpha_1^2}{2}+\frac{12}{13}(t-\tfrac{1}{2})(\tfrac{t^3}{3}-\tfrac{t^2}{2}+\tfrac{1}{12}) -\alpha_1\frac{\cos 2\pi t}{\pi}-t+t^2,$$

$$\int_0^1 K_{12}^1(t,s)^2 ds = \frac{2}{15} + \frac{1987}{5917}\left(t - \frac{1}{2}\right)^2 + \frac{\alpha_1^2}{30} - \frac{t^3}{3} - \frac{2t}{3} + t^2 - \frac{t^4}{12}$$
$$+ \alpha_1\left(\frac{1}{30} - \frac{t}{6} + \frac{t^3}{3} - \frac{t^4}{6}\right) + \frac{1}{13}\left(t - \frac{1}{2}\right)\left(-\frac{49}{15} + \frac{87}{10} + \frac{t^4}{2} - \frac{t^5}{5} + 4t^3 - 13t\right);$$

$$\int_0^1 K_{21}^1(t,s)^2 ds = \frac{12}{1690} + \frac{\alpha_2^2}{2},$$

$$\int_0^1 K_{22}^1(t,s)^2 ds = \frac{1987}{5917} + \frac{\alpha_2^2}{30} + \frac{2t^3}{13} - \frac{t^4}{13} + \frac{12}{13}t^2 + \alpha_2\left(-\frac{1}{6} - \frac{2t^3}{3} + t^2\right) - t.$$

Using the above expressions, one easily gets

$$\theta_1 = \left(\int_0^1 \int_0^1 \sum_{i,j=1}^{2} K_{ij}^1(t,s)^2 ds dt\right)^{1/2} < 0.9 \tag{2.3}$$

and
$$\theta_1 = \max\left(\left(\sup_{t\in[0,1]} \int_0^1 [K_{11}^1(t,s)^2 + K_{12}^1(t,s)^2] ds\right)^{1/2}, \left(\sup_{t\in[0,1]} \int_0^1 [K_{21}^1(t,s)^2 + K_{22}^1(t,s)^2] ds\right)^{1/2}\right)$$
$$< 1.2. \tag{2.4}$$

Here,
$$(G(x))(t) = \begin{pmatrix} 0 \\ -\alpha(1-x_1^2)x_2 + \beta(x_1(t - \frac{1}{4}) + \sin 2\pi t \end{pmatrix}.$$

Note that
$$x(t) = \begin{cases} x(t) & \text{for } t \in [0,1], \\ x(t+1) & \text{for } t \in [-\frac{1}{4}, 0]. \end{cases}$$

Take $m=1, \eta_1 = 5.465463$
$$\begin{pmatrix} \frac{t^4}{12} - \frac{t^3}{6} + \frac{t}{12} - \frac{1}{60} - \frac{\cos 2\pi t}{2\pi} \\ \frac{t^3}{3} - \frac{t^2}{2} + \frac{1}{12} \end{pmatrix}$$

and $S = < \phi_1, \phi_2, \eta_1 >$.

We have $M = 3$. The isomorphism $\Gamma_1 : E^3 \to S_o$ and $\Gamma_2 : <w_1> \to E^1$ are given by $\Gamma_1(\xi) = \Gamma_1(b_1, b_2, c_1)$ $= b_1\phi_1 + b_2\phi_2 + c_1\eta_1$ and $\Gamma_2(u_1 w_1) = u_1$. Obviously, $Q_1 x = (x, w_1)w_1$, $x \in S$ and $Q_1 L \Gamma_1(\xi) = 5.465463\ c_1(\frac{\sin 2\pi t}{t^2 - t})$. Similarly finding $Q_1 G(\Gamma_1(\xi))$ and substituting in $\tilde{\Psi} = \Gamma_2 \Psi \Gamma_1$, we get $\tilde{\Psi}(\xi) = 3.9914098c_1 + \alpha(0.013280934c_1^3 - 0.1794344b_2^2 c_1$

$+ 0.0988283b_1 b_2 c_1 - 0.00000340885b_1^2 c_1 + 0.2192644b_1^2 b_1 - 0.3767367b_2 c_1^2$

$+ 0.01011908b_2^3 - 0.2192644b_2) + \beta(0.2282176b_1 - 0.020556b_2 -$

$-0.0002050636c_1)$. (2.5)

Clearly, we notice that $\xi_o = (0,0,0)$ is a solution of $\tilde{\Psi}(\xi) = 0$. Take $x_o = \binom{0}{0}$. Then

$$||G(x_o)|| \leq \frac{\beta}{\sqrt{2}}. \quad \text{Hence, } ||H(I-Q_1)G(x_o)|| \leq 0.6363962\beta = e \tag{2.6}$$

$$\text{and } \mu(H(I-Q_1)G(x_o)) \leq 0.8485282 = \hat{e}. \tag{2.7}$$

Let $\xi \in E^3$ and $x = \Gamma_1(\xi)$. Then in the usual way we can show that $||x|| = |\xi|$ and $\mu(x) \leq 1.5\ |\xi| = 1.5\ ||x||$. (2.8)

Take the sets $U = \{\xi \in E^3 |\ |\xi - \xi_o| = |\xi| \leq c\}$

and $V = \{x \in S_o |\ ||x - x_o|| = ||x|| \leq c, \mu(x) \leq 1.5c\}$.

We notice that the map $\Gamma_1 : E^3 \to S_o$ maps the ball U continuously into V. We take $\varepsilon = c$ and $r = 1.5c$. Also, for $x, y \in \tilde{S}_o$, we get $|G_1 x - G_1 y| = 0$, $|G_2 x - G_2 y| \leq (|\alpha|^2((1+\hat{R}^2)^2 + 4\hat{R}^4)$

$+ |\beta|^2)^{1/2}(|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_1(t-1/4) - y_1(t-1/4)|^2$

$+ |x_2(t-1/4) - y_2(t-1/4)|^2)^{1/2}$.

Therefore, $\| Gx-Gy \| \leq \sqrt{2}(|\alpha| (1+3\hat{R}^2)+ |\beta|) \|x-y\|$.

Hence, we have

$$k_o = \sqrt{2}(|\alpha| (1+3\hat{R}^2) + |\beta|). \tag{2.9}$$

As we have estimated all the required quantities, we can apply the Theorem 1.7 to get a solution of (2.1). Conditions of Theorem 1.7 are equivalent to

$${}_1k_o < 0.9 \times \sqrt{2}(|\alpha| (1+3\hat{R}^2) + |\beta|) < 1,\ 0 < c < d,\ 0 < 1.5c < \hat{R},$$

$$c+e < c + 0.636397\beta \leq (1-0.9 \times \sqrt{2}(|\alpha| (1+3\hat{R}^2)+ |\beta|))d < (1-\theta_1 k_o)d,$$

$$r+\hat{e} < 1.5c +0.848529\beta \leq \hat{R}-1.2 \times \sqrt{2} (|\alpha| (1+3\hat{R}^2)+ |\beta|)d < \hat{R}-\theta_1 k_o d,$$

$$(\theta_1 k_o d+e)k_o \leq (0.9 \times \sqrt{2}(1+3R^2) + |\beta|)d+0.636397 \beta)$$

$$\times \sqrt{2} (|\alpha| (1+3R^2) + |\beta|) \leq \delta ,$$

$\bar{\psi}(\xi_-) \geq \delta$ and $\bar{\psi}(\xi_2) \leq -\delta$ where $\delta > 0$ and $\bar{\psi}$ is given by (2.5).

We can easily check that one possible choice for the above quantities is

$c = 0.01$, $d = 0.15$, $R = 0.2$, $|\alpha| \leq 0.1$, $|\beta| \leq 0.1$, and

$$\xi_1 = (0,0,0.01) \text{ and } \xi_2 = (0,0,-0.01).$$

Hence by Theorem 1.7, the problem (2.2) and therefore the problem (2.1) has atleast one solution x. Moreover, $|x| \leq 0.2$.

CHAPTER-3

1. Introduction :

In the present chapter we consider abstract operational equations of the form $Lx = Nx$ and $Lx + \alpha Ax = Nx$ where $L : D(L) \subseteq X \to Y$, $N : D(N) \subseteq X \to Y$, $A : X \to Y$ are operators, L linear, not necessarily bounded, N and A continuous not necessarily linear, X and Y Banach spaces and α a real parameter. On the continuous operator A we assume that it maps bounded sets to bounded sets and on the operator N we assume that it satisfies certain growth conditions in the large.

We prove in terms of the alternative method and the Leray-Schauder topological degree that under such assumptions on N and A, there exist solutions to $Lx = Nx$ and $Lx + \alpha Ax = Nx$ for $|\alpha| < \alpha_o$ for some $\alpha_o > 0$ and moreover the solutions remain uniformly bounded as α is allowed to pass through the point of resonance $\alpha = 0$.

2. Notations and Preliminaries :

Let X and Y be Banach spaces. $P : X \to X$, $Q : Y \to Y$ be projection operators (linear, bounded and idempotent) such that

$$R(P) = P(X) = X_o \qquad R(Q) = Q(Y) = Y_o$$

$$\text{Ker } P = (I-P)X = X_1 \qquad \text{Ker } Q = (I-Q)Y = Y_1$$

Let $\text{Ker } L = X_o$, $R(L) = Y_1$, $1 \leq \dim \text{Ker } L = \dim X_o < +\infty$.

Let there be a linear operator H so that the following conditions are satisfied.

(i) $H(I-Q)Lx = (I-P)x$ for all $x \varepsilon D(L)$.

(ii) $QLx = LPx$ for all $x \varepsilon D(L)$.

(iii) $LH(I-Q)Nx = (I-Q)Nx$ for all $x \varepsilon D(N)$.

Under these assumptions the equation $Lx = Nx$ can be decomposed into an equivalent system of equations. (see [8]).

$$x = Px + H(I-Q)Nx \tag{3.1}$$

$$QNx = 0. \tag{3.2}$$

The first of these equations can be called the auxiliary equation and the second the bifurcation equation. Because of conditions (i) and (iii) H is seen to act like a right partial inverse for L and indeed we make the assumptions $H: R(L) \to D(L) \cap X_1$ is a compact operator. We now assume that Y is a space of linear operators on X so that the operation $\langle y,x \rangle : YxX \to R$ is defined, is linear in both x and y and $|\langle y,x \rangle| \leq K\,||x||_X\,||Y||_Y$ for some constant $K > 0$ and all $x \varepsilon X$ and $y \varepsilon Y$. As an example let $X = Y = L_2(G)$ where G is a bounded open set in R^n, then $|\langle y,x \rangle| = |\int_G x(t)y(t)dt| \leq ||x||_X\,||Y||_Y$. The operation $\langle y,x \rangle$ has the following property. For $y \varepsilon Y$, $Qy = 0$ if and only if $\langle Qy,x^* \rangle = 0$ for all $x^* \varepsilon X_o$. Let $||P|| = K$, $||I-P|| = K'$, $||Q|| = \chi$, $||(I-Q)|| = \chi'$, $||H|| = M$. Whenever X and Y are Hilbert spaces and P and Q are orthogonal projections, then $K = K' = \chi = \chi' = 1$.

3. $Lx = Nx$.

Theorem 1. (Case of N bounded) : Let X and Y be Banach spaces. Let L,H,P,Q satisfy the conditions stated in section 2 of the present chapter. Let $N : X \to Y$ be a continuous operator and let X_0 = Ker L be finite dimensional. Suppose $J:R(Q) \to R(P)$ be an isometric isomorphism. If (a) there exist a constant J_0 such that $||Nx|| \leq J_0$ for all $x \in X$ and (b) there is a real number $R_0 \geq 0$ such that $< JQNx, x^* > \leq 0$ [or $<JQNx, x^*> \geq 0$] for all $x \in X$, $x^* \in X_0$ with $Px = x^*$, $||Px|| = R_0$, $|| x-x^* || \leq M\chi' J_0$ then the equation $Lx = Nx$ has at least one solution in D(L).

Proof. Assuming the conditions stated above we have

$$
\begin{aligned}
Lx = Nx \quad &\Longleftrightarrow \quad Q(Lx-Nx) = 0 \;;\; (I-Q)(Lx-Nx) = 0. \\
&\Longleftrightarrow \quad QNx = 0 \;;\quad H(I-Q)Lx = H(I-Q)Nx. \\
&\Longleftrightarrow \quad JQNx = 0 \;;\quad (I-P)x = H(I-Q)Nx. \\
&\Longleftrightarrow \quad JQNx = 0 \;;\quad x = Px + H(I-Q)Nx. \\
&\Longleftrightarrow \quad x = Px + H(I-Q)Nx + JQNx. \qquad (3.3)
\end{aligned}
$$

In the last statement one way implication is obvious. In order to see the other way, let $x = Px + H(I-Q)Nx + JQNx$. Operating on both sides by P we get $Px = Px + 0 + JQNx \Rightarrow JQNx = 0$. Thus instead of the equation $Lx + Nx$, we shall consider the equation $x = Px + H(I-Q)Nx + JQNx$. Let D denote the set $\{x \in X.\ ||Px|| \leq R_0, ||x|| \leq C_0\}$ where C_0 is a constant to be chosen subsequently.

Now there are two possibilities. Either (a) there is a solution of (3.3) on ∂D (the boundary of D) in which case the theorem is true or (b) there is no solution of (3.3) on ∂D. We consider this second case below.

Define $L_\lambda(x) = x - \lambda(Px + H(I-Q)Nx + JQNx) = (I - G_\lambda)x$. Clearly $L_0 x = Ix$ and $L_1 x = x - Px - H(I-Q)Nx - JQNx$. We show that in the closed convex set D the degree of the map L_λ with respect to D and the image $x = 0$ is defined i.e. $d(L_\lambda, D, 0)$ is defined. Further $d(L_\lambda, D, 0)$ is constant for $0 \leq \lambda \leq 1$.

To see this we need only to show that

$$x = \lambda(Px + H(I-Q)Nx + JQNx). \tag{3.4}$$

has no solution on ∂D. In order that $x \in \partial D$ either (i) $||Px|| = R_0$ and $||x|| \leq C_0$ or (ii) $||Px|| \leq R_0$ and $||x|| = C_0$ must hold. In the latter case if (3.4) has a solution, then $C_0 = ||x|| \leq ||Px|| + M\chi' J_0 + \chi J_0 \leq R_0 + M\chi' J_0 + \chi J_0$, which is a contradiction if we choose $C_0 > R_0 + M\chi' J_0 + \chi J_0$. In case (3.4) has a solution with $||Px|| = R_0$ and $||x|| \leq C_0$ for some $0 < \lambda < 1$, then again we show that we arrive at a contradiction. For $\lambda = 1$ we have already assumed that (3.4) does not have a solution on ∂D. On operating by P on both sides of (3.4) we have $Px = \lambda Px + \lambda JQNx$. Substituting in equation (3.4) we get $x - Px = \lambda H(I-Q)Nx$ so that $||x - Px|| < M\chi' J_0$. Thus denoting $Px = x^*$ we have $||x - x^*|| < M\chi' J_0$ and $||Px|| = R_0$. So $\langle JQNx, x^* \rangle \leq 0$ by assumption. Now taking bracket operation on the equation $(1-\lambda)Px = \lambda JQNx$,

we have $(1-\lambda) < Px, Px > \; = \; \lambda < JQNx, Px > \; \leq 0$. But $(1-\lambda) < Px, x > \; > 0$.

This contradiction establishes that (3.3) does not have a solution for any $x \in \partial D$. Thus $d(I-G_1, D, 0) = d(I, D, 0) \neq 0$. Hence $x = Px + H(I-Q)Nx + JQNx$ has a solution. This completes the proof of the theorem.

We remark here that if we had considered the case where $< JQNx, x^* > \; \geq 0$, we could have taken $-J$ in place of J.

<u>Theorem 1^*</u>. Let X and Y be Banach spaces. Let L.H.P,Q satisfy the conditions stated in Section 2. Let N be a continuous mapping from X to Y. Let X_0, the kernel of L be finite dimensional. Suppose $J : R(Q) \to R(P)$ be an isometric isomorphism. Let there exist a constant J_0 such that $|| Nx || \leq J_0$ for all $x \in S$ where S is an open set in X. Suppose that $< JQNx, x^* > \; \leq 0$ [or $<JQNx, x^*> \; \geq 0$] for all $x \in S$, $x^* \in X_0$ with $Px = x^*$, $|| Px || = R_0$, $|| x - x^* || \leq M' J_0$. Suppose also that the equation $x = \lambda(Px + H(I-Q)Nx + JQNx)$ has a solution for some $\lambda \in (0,1)$ only if $|| Px || = R_0$ or $|| x || = C_0$, $C_0 > R_0$, then the equation $Lx = Nx$ has at least one solution in D(L).

<u>Proof</u>. The proof is similar to that of Theorem 1.

A theorem similar to the above is valid when N has a limited growth as given below.

<u>Theorem 2</u>. (Case of limited growth of N). Let $\phi(\xi) \geq 0$, $\psi(\xi) \geq 0$, $0 \leq \xi < +\infty$ be monotone non-decreasing functions. Let

us assume that $\| Nx \| \leq \phi(\| x \|)$ for all $x \in X$, and $\langle JQNx, x^* \rangle \leq 0$ [or $\langle JQNx, x^* \rangle \geq 0$] for all $x \in X$, $x^* \in X_o$ with $Px = x^*$, $\|x^*\| = R_o$, $\| x - x^* \| \leq \Psi(\|x\|)$ where $\Psi(\xi) \geq M\gamma' \phi(\xi)$. Further let there be a constant $C_o > R_o + (M\gamma' + \epsilon)\phi(C_o)$. Then the equation $Lx = Nx$ has at least one solution $x \in D(L) \cap X$ with $\|x\| \leq C_o$ and $\| Px \| \leq R_o$.

<u>Proof</u>. As we had seen earlier we need to prove the existence of a solution of $x = Px + H(I-Q)Nx + JQNx$. As in Theorem 1, we choose $D = \{ x \in X : \|Px\| \leq R_o, \|x\| \leq C_o \}$. We suppose that (3.3) does not have a solution on ∂D. Otherwise, the theorem is proved. As in the previous theorem we consider the operator $L_\lambda x = (I - G_\lambda)x = x - \lambda(Px + H(I-Q)Nx + JQNx)$. Obviously $L_1 x \neq 0$ on ∂D (by assumption). We show that $L_\lambda x = 0$ does not have a solution for $\lambda \in (0,1)$, $x \in \partial D$. Suppose contrary, then there exist $\lambda_1 \in (0,1)$ and $x_1 \in \partial D$ such that $L_{\lambda_1} x_1 = 0$. We consider the case (a) where $\|Px_1\| = R_o$ and $\|x_1\| \leq C_o$. Then we have

$$x_1 = \lambda_1 Px_1 + \lambda_1 H(I-Q)Nx_1 + \lambda_1 JQNx_1 \tag{3.5}$$

Applying projection operator P on both sides of (3.5) we have

$$Px_1 = \lambda_1 Px_1 + \lambda_1 JQNx_1 \tag{3.6}$$

Substituting this equation in (3.5) we get $x_1 - Px_1 = \lambda H(I-Q)Nx_1$ so that

$$\| x_1 - Px_1 \| = \| \lambda_1 H(I-Q)Nx_1 \| \leq M\gamma' \phi(\| x_1 \|) \leq \Psi(\| x_1 \|).$$

But carrying out the bracket operation on both sides of (3.6)

we get

$$0 < (1-\lambda_1)\langle Px_1, Px_1\rangle = \lambda_1 \langle JQNx_1, Px_1\rangle \leq 0$$

which is a contradiction. Consider now case (b) where $L_{\lambda_1}x_1 = 0$ with $\|x_1\| = C_0$ and $\|Px_1\| \leq R_0$. In this case

$$\begin{aligned} \|x_1\| &\leq \lambda_1 \|Px_1\| + \lambda_1 \|H(I-Q)Nx_1\| + \lambda_1 \|JQNx_1\| \\ &\leq R_0 + M\gamma'\phi(\|X_1\|) + \gamma\phi(\|x_1\|). \\ &= R_0 + (M\gamma'+\gamma)\phi(C_0). \end{aligned}$$

So $C_0 \leq R_0+(M\gamma'+\gamma)\phi(C_0)$ which is again a contradiction.

This completes the proof of the theorem.

<u>Theorem 2*</u> : (Case of limited growth of N). Let $\phi(\xi) \geq 0$, $\Psi(\xi) \geq 0$, $0 \leq \xi < +\infty$ be monotone non-decreasing functions. Let us assume that $\|Nx\| \leq \phi(\|x\|)$ for all $x \in S$ where S is an open set in X and $\langle JQNx, x^*\rangle \leq 0$ [or $\langle JQNx, x^*\rangle \geq 0$] for all $x \in S$, $x^* \in X_0$. $Px = x^*$, $\|Px\| = R_0$, $\|x-x^*\| \leq \Psi(\|x\|)$, where $\Psi(\xi) \geq M\gamma'\phi(\xi)$. Let $C_0 > R_0 + (M\gamma' + \gamma)\phi(C_0)$. Further assume that the equation $x = \lambda(Px + H(I-Q)Nx + JQNx)$ has a solution for some $\lambda \in (0,1)$ unless either $\|Px\| = R_0$ or $\|x\| = C_0$, then the equation $Lx = Nx$ has at least one solution $x \in D(L)$ with $\|x\| \leq C_0$ and $\|Px\| \leq R_0$.

<u>Proof</u>. Proof is similar to that of Theorem 2.

We now consider abstract operational equations of the form $Lx + \alpha Ax = Nx$ where $L : D(L) \subseteq X \to Y$, $N:X \to Y$ and $A:X \to Y$. Here L is a linear operator not necessarily bounded and N and

A are continuous operators not necessarily linear. It is also assumed that A maps bounded sets into bounded sets. Following Cesari we prove that the conditions imposed on N and L actually have stronger implications, namely that under such assumptions there are numbers $\alpha_o > 0$, $C_o > 0$ such that for all real α with $|\alpha| \leq \alpha_o$ the equation $Lx + \alpha Ax = Nx$ has at least one solution $x \in X$ with $\| x \| \leq C_o$. Stating otherwise the parameter α can pass through the point of resonance $\alpha = 0$ and yet uniformly bounded solutions x can be guaranteed. As pointed out by Cesari this has a physical significance in the sense that for the case of the periodic solutions of ordinary differential systems with forcing terms of a given period there exist uniformly bounded periodic solutions of the same period.

4. $Lx + Ax = Nx$.

<u>Theorem 3</u>. (Existence across a point of resonance in case of N bounded).

Let the general conditions of Theorem 1 be satisfied and $A : X \rightarrow Y$ be a continuous and bounded operator. Suppose (a) there is a constant J_o such that $\| Nx \| \leq J_o$ for all $x \in X$, (b) there are constants $R_o > 0$, $\theta > 0$, $\beta > M J_o$ such that $\langle JQNx, x^* \rangle < -\theta \| x^* \|$ [or $\langle JQNx, x^* \rangle > \theta \| x^* \|$] for all $x \in X$, $x^* \in X_o$ with $Px = x^*$, $\| x^* \| = R_o$, $\| x - x^* \| \leq \beta$, then there are also constants $\alpha_o > 0$, $C_o > 0$ such that for every α with $|\alpha| \leq \alpha_o$, the equation $Lx + \alpha Ax = Nx$ has

at least one solution $x \varepsilon D(L)$ with $||x|| \leq C_0$.

Proof. The proof follows the pattern of previous theorems. Let $D = \{ x \varepsilon X : ||x|| \leq C_0, || Px|| \leq R_0 \}$. We first show that there is no solution of

$$x = \lambda\lfloor Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)Nx - JQAx\rfloor \qquad (3.7)$$

for $\lambda \varepsilon (0,1)$ and $|\alpha| \leq \alpha_0$ for some α_0 with $||Px|| = R_0$ and $||x|| \leq C_0$. Suppose contrary, then there exist a $\lambda_1 \varepsilon (0,1)$ and x_1 with $||Px_1|| = R_0$ such that

$$x_1 = \lambda_1\lfloor Px_1 + H(I-Q)Nx_1 + JQNx_1 - \alpha H(I-Q)Nx_1 - \alpha JQAx_1\rfloor \qquad (3.8)$$

Operating by P on both sides, we get

$$Px_1 = \lambda_1\lfloor Px_1 + JQNx_1 - \alpha JQAx_1] \qquad (3.9)$$

Substituting this equation in (3.8), we get

$$x_1 - Px_1 = \lambda_1\lfloor H(I-Q)Nx_1 - \alpha H(I-Q)Ax_1]$$

so that $|| x_1 - Px_1|| = ||\lambda_1\lfloor H(I-Q)Nx_1 - \alpha H(I-Q)Ax_1\rfloor ||$

$$\leq M' J_0 + |\alpha| M' B.$$

where $||Ax|| \leq B$ for all $x \varepsilon D$. Choosing α_0 to be sufficiently small, we see that $\mu' J_0 + |\alpha| M' B < \beta$. Thus $|| x_1 - Px_1|| < \beta$. Now carrying out the bracket operation on both sides of (3.9) we get

$$0 \leq (1-\lambda)(Px_1, Px_1) = \lambda < JQNx_1 - \alpha JQAx_1, Px_1 > .$$
$$< \lambda\lfloor - \theta ||Px_1|| - \alpha < JQAx_1, Px_1 >].$$
$$\leq \lambda\lfloor - \theta R_0 + \alpha B R_0].$$

Now choosing α_0 to be sufficiently small, we can ensure that $\alpha_0 \chi B R_0 \leq \frac{\lambda}{2}\theta' R_0$. So $(1-\lambda_1)\langle Px_1, Px_1\rangle \leq -\frac{\lambda}{2}\theta R_0$ contradicting the fact that $\langle Px_1, Px_1\rangle \geq 0$. This contradiction shows that $x = \lambda[Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)Ax - \alpha JQAx]$ does not have solution with $||Px|| = R_0$ and $||x|| \leq C_0$ for $\lambda \in (0,1)$ and $|\alpha| \leq \alpha_0$ for small α_0.

Now we show that (3.8) does not have a solution with $||x|| = C_0$ and $||Px|| \leq R_0$ for all $\lambda \in (0,1)$ for $|\alpha| \leq \alpha_0$ where α_0 is small. Suppose contrary, then $||x_1|| \leq \lambda_1[||Px_1|| + M\chi' J_0 + \chi J_0 + \alpha\chi B + \alpha M\chi' B]$ or $C_0 \leq R_0 + M\chi' J_0 + \chi J_0 + \alpha[\chi B + M\chi' B]$. We choose $C_0 = R_0 + M\chi' J_0 + \chi J_0 + 1$ and $|\alpha_0| \leq (\chi B + M\chi' B)^{-1}$. Then obviously $C_0 > R_0 + M\chi' J_0 + \chi J_0 + \alpha(\chi B + M\chi' B)$ for all $|\alpha| \leq \alpha_0$ which is again a contradiction. Thus for a suitable choice of α_0 and C_0, (3.8) does not have a solution on ∂D for $\lambda \in (0,1)$ and $|\alpha| \leq \alpha_0$.

Finally for any α with $|\alpha| \leq \alpha_0$, either $x = Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)Ax - \alpha JQNx$ has a solution on ∂D in which case the theorem is true or it does not have a solution. In case $x = \lambda(Px + H(I-Q)Nx + JQNx - \alpha JQAx - \alpha H(I-Q)Ax)$ does not have a solution on ∂D for any $\lambda \in [0,1]$, $d(I-G_\lambda, D, 0)$ is constant for all $\lambda \in [0,1]$ where $G_\lambda = \lambda[Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)Ax - \alpha JQAx]$. Obviously $d(I-G_0, D, 0) \neq 0$. So $(I-G_\lambda)x = 0$ has a solution in D. This completes the proof.

<u>Theorem 3*</u>. (Existence across a point of resonance in case of N bounded).

Let the general conditions of Theorem 1 be satisfied and let $\Lambda . X \to Y$ be a continuous and bounded operator. Suppose (a) there is a constant J_o such that $||Nx|| \leq J_o$ for all $x \in S$, where S is an open set, (b) there are constants $R_o > C$ $\theta > 0$, $\beta > M' J_o$ such that $\langle JQNx, x^* \rangle < -\theta ||x^*||$ $[$or $\langle JQNx, x^* \rangle > \theta ||x^*||]$ for all $x \in S$, $x^* \in X_o$ with $Px = x^*$, $||x^*|| = R_o$, $||x-x^*|| \leq \beta$. Further assume that the equation. $x = \lambda[Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)\Lambda x - \alpha JQ\Lambda x]$ has a solution for some $\lambda \in (0,1)$ unless either $||Px|| = R_o$ or $||x|| = C_o$ where C_o is a subsequently chosen suitable constant. Then there is a constant $\alpha_o > 0$ such that for every α with $|\alpha| \leq \alpha_o$, the equation $Lx + \alpha\Lambda x = Nx$ has at least one solution $x \in D(L)$ with $||x|| \leq C_o$.

<u>Proof</u>. The proof is similar to that of Theorem 3.

<u>Theorem 4</u>. (Existence across a point of resonance in case of limited growth of N).

Under the general conditions of Theorem 2 let $A:X \to Y$ be a continuous bounded operator. Let ϕ, Ψ, ϕ_1 be monotonically decreasing non-negative real valued functions on $[0,\infty]$ with both $\phi_1(\xi)$ and $\Psi(\xi)$ positive for $\xi = R_o$. Suppose (a) $||Nx|| \leq \phi(||x||)$ for all $x \in X$ (b) $\langle JQNx, x^* \rangle \leq -\phi_1(||x^*||)$ $[$ or $\langle JQNx, x^* \rangle \geq \phi_1(||x^*||)$ $]$ for all $x \in X$, $x^* \in X_o$. with $Px = x^*$, $||x^*|| = R_o$, $||x-x^*|| \leq \Psi(||x||)$ and further (c) let there be a constant C_o with $C_o > R_o+(M\chi' + \chi)\phi(C_o)$,

$M\chi'\phi(\xi) < \Psi(\xi)$ then there is $\alpha_o > 0$ such that for every real α with $|\alpha| \leq \alpha_o$, the equation $Lx + \alpha Ax = Nx$ has at least a solution $x \in D(L)$ with $||x|| \leq C_o$.

<u>Proof</u>. First we show that

$$x = \lambda[px + H(I-Q)Nx + JQNx - \alpha H(I-Q)Ax - \alpha JQAx] \tag{3.10}$$

does not have a solution on ∂D where $D = \{x : ||Px|| \leq R_o, ||x|| \leq C_o\}$. Suppose contrary then there is a solution on ∂D. Suppose x_1 is such a solution with $|x_1|| \leq C_o$, $||Px_1|| = R_o$.
Then $Px_1 = \lambda[Px_1 + JQNx_1 - \alpha JQAx_1]$ (3.11)

and $x_1 - Px_1 = \lambda[H(I-Q)Nx_1 - \alpha H(I-Q)Ax_1]$.

Thus $||x_1 - Px_1|| \leq M\chi'[\phi(||x_1||) + \alpha B] < \Psi(||x_1||)$.
for sufficiently small α. Taking the bracket operation on both sides of (3.11) with respect to Px_1 we have

$$\begin{aligned} 0 \leq (1-\lambda)\langle Px_1, Px_1\rangle &\leq \lambda[\langle JQNx_1, Px_1\rangle - \alpha\langle JQAx_1, Px_1\rangle]. \\ &\leq -\lambda[\phi_1(||x_1^*||) - \alpha\chi\phi(||x_1^*||)||Px_1||]. \\ &\leq -\lambda[\phi_1(R_o) - \alpha\chi\phi(C_o)R_o] < 0. \end{aligned}$$

for sufficiently small α. This contradiction shows that such a solution cannot exist. Now we consider the other possibility of having a solution of (3.10) on ∂D with $||Px|| \leq R_o$ and $||x|| \leq C_o$. Suppose x_2 is such a solution.
Then

$$\begin{aligned} C_o = ||x_2|| &\leq \lambda[||Px_2|| + ||H(I-Q)Nx_2|| + ||JQNx_2|| \\ &\quad + |\alpha|\,||H(I-Q)Nx_2|| + |\alpha|\,||JQAx_2||] \\ &\leq \lambda[R_o + (M\chi' + \chi)\phi(C_o) + |\alpha|(M\chi'\phi(C_o) + \chi B)] \end{aligned} \tag{3.12}$$

Where $||\Lambda x|| \leq B$ for all $x \varepsilon D$. Since by hypothesis $C_o > R_o + (M\gamma' + \gamma)\phi(C_o)$, it is clear that (3.12) is violated for sufficiently small α. This contradiction shows that $x = \lambda[Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)Nx - \alpha JQ\Lambda x]$ does not have a solution for $\lambda \varepsilon [0,1]$ for small α.

Now coming to the final arguments of the proof we have either a solution of $x = Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)\Lambda x - \alpha JQ\Lambda x$ on ∂D and hence the bifurcation equation has a solution or it does not have a solution on ∂D. In the latter case, $x = \lambda[Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)\Lambda x - \alpha JQ\Lambda x]$ does not have a solution for $\lambda \varepsilon [0,1]$. So $d(I-G_\lambda, D, 0) = d(I, D, 0) = d(I-G_1, D, 0) \neq 0$ for small α where $G_\lambda = \lambda[Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)\Lambda x - \alpha JQ\Lambda x]$. Hence $(I-G_1)x = 0$ has a solution in D. This completes the proof.

<u>Theorem 4*</u>. (Existence across a point of resonance in case of limited growth of N).

Under the general conditions of Theorem 2 let $A : X \to Y$ be a continuous bounded operator. Let ϕ, Ψ, ϕ_1 be monotonically decreasing non-negative real valued functions on $[o, \infty]$ with both $\phi_1(\xi)$, $\Psi(\xi)$ positive for $\xi = R_o$. Suppose (a) $||Nx|| \leq \phi(||x||)$ for all $x \varepsilon S$, where S is an open set (b) $\langle JQNx, x^* \rangle \leq [-\phi_1(||x^*||)]$ or $[\langle JQNx, x^* \rangle \geq \phi_1(||x^*||)]$ for all $x \varepsilon S$, $x^* \varepsilon X_o$ with $Px = x^*$, $||x^*|| = R_o$, $||x-x^*|| \leq \Psi(||x||)$ and further (c) let there be a constant

C_o with $C_o > R_o + (M\gamma' + \varepsilon)\phi(C_o)$, $M\gamma'\phi(\xi) < \Psi(\xi)$. suppose also that the equation $x = \lambda[Px + H(I-Q)Nx + JQNx - \alpha H(I-Q)\Lambda x - \alpha JQ\Lambda x]$ has a solution for some $\lambda \in (0,1)$ unless either $||Px|| = R_o$ or $||x|| = C_o$, then there is a constant $\alpha_o > 0$ such that for every real α with $|\alpha| \leq \alpha_o$, the equation $Lx + \alpha\Lambda x = Nx$ has at least a solution $x \in D(L)$ with $||x|| \leq C_o$.

<u>Proof</u>. The proof is similar to Theorem 4.

As an example for our theory developed, we consider the following delay model for single species population growth. In the growth model considered below we utilize two parameters, the inherent species growth rate, and the 'magnitudes' of the delayed and non-delayed growth rate response to density changes. We consider the scalar functional equation

$$x'(t) = -\lambda_1 x(t) - \lambda_2 \int_0^\infty x(t-s)dh(s) + g(x,\lambda_1,\lambda_2,t) \qquad (N)$$

where λ_1 and λ_2 are reals and $\int_0^\infty |dh(s)| = 1$. We will make specific assumptions about the integrator $h(s)$ and the functional g.

We study the linear problems

$$y'(t) = -\lambda_1 y(t) - \lambda_2 \int_0^\infty y(t-s)dh(s) \qquad (H)$$

$$y'(t) = -\lambda_1 y(t) - \lambda_2 \int_0^\infty y(t-s)dh(s) - f(t) \qquad (NH)$$

where $f(t) \in P(p)$, the Banach space of continuous p-periodic functions under the supremum norm $||f||_\infty = \sup_{0\leq t\leq p} |f(t)|$. The adjoint to equation (H) is given by

$$z'(t) = \lambda_1 z(t) + \lambda_2 \int_0^\infty dh(s)z(t+s) \qquad (A)$$

We show that the adjoint indeed takes this form. We show that $\langle Lx, z \rangle = \langle x, L^* z \rangle$ where the operator equation for L and L^* are given respectively by

$$y'(t) = -\lambda_1 y(t) - \lambda_2 \int_0^\infty y(t-s)dh(s) \qquad (H)$$

$$z'(t) = \lambda_1 z(t) + \lambda_2 \int_0^\infty dh(s)z(t+s) \qquad (A)$$

Define,

$$\Lambda x = x'(t) + \lambda_1 x(t) + \lambda_2 \int_0^\infty x(t-s)dh(s).$$

We want to solve $\langle \Lambda x, z \rangle - \langle x, G \rangle = 0$ for G. Or,

$$\int_0^p \{ z^*(t)x'(t) + \lambda_1 z^*(t)x(t) + \lambda_2 z^*(t)\int_0^\infty x(t-s)dh(s) - G^*(t)x(t) \} dt = 0 \qquad (3.13)$$

since $\displaystyle\int_0^\infty x(t-s)dh(s) = -\int_t^{-\infty} x(u)d_u h(t-u) = \int_{-\infty}^t x(s)d_s h(t-s),$

We obtain

$$\int_0^p \{ z^*(t)x'(t) + \lambda_1 z^*(t)x(t) + \lambda_2 z^*(t)\int_{-\infty}^t x(s)d_s h(t-s) - G^*(t)x(t) \} dt = 0 \qquad (3.14)$$

Since h(s) is assumed to be of bounded variation defined on a suitable set and $x(s) \varepsilon P(p)$, the Banach space of p-periodic functions, we have

$$\int_{-\infty}^t x(s)d_s h(t-s) = \left[\int_{t-p}^t + \int_{t-2p}^{t-p} + \dots + \int_{t-np}^{t-(n-1)p} \dots \right] (x(s)d_s h(t-s))$$

$$= \int_{t-p}^t x(s)d \, \Sigma \, h(t-s) + h(t-s+p) + h(t-s+2p) + \dots$$

$$= \int^t x(s)dH(t-s).$$

where $dH(t-s) = d\,\Sigma\, h(t-s) + h(t-s+p) + h(t-s+2p)+\ldots$

$$= \int_0^p x(s)\, d_s H^*(t-s,t).$$

Here we assume that H exists and is a function of bounded variation for each t. We can change the domain of integration of the double integral (see [38])

$$\int_0^p z\,(t) \int_{-\infty}^{t} x(s)d_s h(t-s)dt = \int_0^p\int_0^p z^*(t)\,[d_s H^*(t-s,t)]x(s)dt.$$

Using an unsymmetric Fubini type theorem due to Cameron and Martin (see [4]) we may interchange the order of integration to obtain

$$\int_0^p\int_0^p z^*(t)[d_s H^*(t+s)]x(s)dt = d_s(\int_0^p z^*(t)H^*(t-s,t)dt)x(s).$$

Substitution of this integral into (3.14) implies

$$\int_0^p z^*(s)x'(s) + \lambda_1 z^*(s)x(s) + \lambda_2 d_s[\int_0^p z^*(t)H^*(t-s)dt]x(s)$$
$$- G^*(s)x'(s)\,\}\,ds = 0.$$

We now integrate the last three terms by parts and use the fact that $x \quad \overset{\circ}{H}{}^{\circ}$ i.e. $\{x \quad L_2[0,p] : x(0) = x(p) = 0\}$ to obtain

$$\int_0^p[z^*(s) -\lambda_1 \int_0^s z^*(t)dt -\lambda_2 \int_0^p z^*(t)H\,(t-s)dt +$$
$$+\int_0^s G^*(t)dt]x'(s)ds = 0.$$

The fundamental lemma of calculus of variation yields (since $\overset{\circ}{H}$ is dense in $L_2[0,p]$).

$$z^*(s)-\lambda_1 \int_0^s z^*(t)dt-\lambda_2 \int_0^p {}^*(t)H\,(t-s,t)dt + \int_0^s G^*(t)dt = \alpha\,.$$

Hence $G^*(s) = -\frac{d}{ds}[z^*(s)-\lambda_1\int_0^s z^*(t)dt-\lambda_2 \int_0^p {}^*(t)H\,(t-s,t)dt]$

or assuming differentiability of $z^*(s)$,

$$G(s) = A^*(z(s)) = -z'(s) + \lambda_1 z(s) + \lambda_2 \int_0^p d_s H^*(t-s,t) z(t) dt$$

Hence the adjoint equation is given by

$$\begin{aligned} z'(s) &= \lambda_1 z(s) + \lambda_2 \int_0^p d_s H^*(t-s,t) z(t) dt \\ &= \lambda_1 z(s) + \lambda_2 \int_0^\infty d_s h(s) z(t+s) dt \end{aligned}$$

which is what was to be proved.

We have the following lemma.

Lemma 1. (Fredholm's alternative).

Let λ_1, λ_2, p be fixed reals with $p > 0$.

(a) The homogeneous problem (H) possesses at most a finite number $k \geq 0$ of linearly independent non-trivial p-periodic solutions $y_i(t) \in P(p)$. The adjoint equation (A) possesses exactly the same number of solutions $z_i(t)$.

(b) If $k = 0$, then the non-homogeneous equation (NH) possesses a unique solution $\bar{y}(t) \in P(p)$ for every $f(t) \in P(p)$.

(c) If $k > 0$ then (NH) has a solution $\bar{y}(t) \in P(p)$ if and only if $f(t) \in P_0(p)$ where $P_0(p) = \{ f \in P(p) : (f, z_i) = p^{-1} \int_0^p f(t) z_i(t) dt = 0 \quad 1 \leq i \leq k$ for all solutions z_i of the adjoint equation (A)$\}$.

Proof. (a) If the complex Fourier series

$$y(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in2\pi p^{-1} t} \qquad \alpha_n = \bar{\alpha}_{-n}, n > 0 \tag{3.15}$$

is substituted into the homogeneous equation (H) and coefficients of like terms are equated, we obtain the equations

$$(\lambda_1+\lambda_2 C_n(p) + i(2\pi np^{-1} - \lambda_2 S_n(p))\alpha_n = 0 \; ; \; n \geq 0 \qquad (3.16)$$

where $C_n(p) = \int_0^\infty \cos 2n\pi p^{-1}s \, dh(s)$.

$$S_n(p) = \int_0^\infty \sin 2n\pi p^{-1}s \, dh(s).$$

Now λ_1 and λ_2 are fixed reals and $C_n(p)$ and $S_n(p)$ are bounded independently of n, hence

$$|(\lambda_1+\lambda_2 C_n(p)+i(2\pi np^{-1}-\lambda_2 S_n(p)))| \to \infty \text{ as } n \to \infty.$$

Hence there exists an n_1 such that $n > n_1$ implies

$$|(\lambda_1+\lambda_2 C_n(p) + i(2\pi np^{-1}-\lambda_2 S_n(p)))| > 0$$

in which case the corresponding α_n s are zero. Since corresponding to each α_n, we have two linearly independent solutions namely, $\cos 2\pi np^{-1}t$ and $\sin 2\pi np^{-1}t$ there are only finitely many non-trivial p-periodic solutions of the homogeneous equation.

If the Fourier series $\sum_{n=-\infty}^{\infty} \alpha_n e^{in2\pi p^{-1}t}$ is substituted into the adjoint equation (A) we get the system of equations

$$(\lambda_1+\lambda_2 C_n(p)-i(2\pi np^{-1}-\lambda_2 S_n(p)))\alpha_n = 0 \qquad (3.17)$$

where the coefficients of α_n are just the conjugate transpose of that of the coefficients for the homogeneous equation.

Hence whenever

$$|(\lambda_1+\lambda_2 C_n(p)+i(2\pi np^{-1}- \lambda_2 S_n(p)))| > 0$$

(c) Let $N = \{ n : |\lambda_1+\lambda_2 C_n(p)+i(2\pi np^{-1}-\lambda_2 S_n(p))| = 0 \}$

This part follows straight forwardly from the fact that (3.18) has a solution α_n for $n \varepsilon N$ if and only if $f_n\alpha_n = 0$ for all solution α_n, $n \varepsilon N$ of the adjoint system (3.17) and that in this case, there is a unique solution α_n orthogonal to the solution space of (3.16) which satisfies $n |\alpha_n| \leq k |f_n|$ $n \varepsilon N$. We deduce the absolute continuity in the same way as in (b).

This proves the lemma.

We are interested in those values of λ_i for which the homogeneous equation (H) and therefore the adjoint equation (A) possess exactly two linearly independent non-trivial p-periodic solutions.

Coming back to equation (3.16)

$$(\lambda_1+\lambda_2 C_n(p) + i(2\pi np^{-1}-\lambda_2 S_n(p)))\alpha_n = 0 \; ; \; n \geq 0.$$

Since non-trivial solutions are obtained if and only if at least one coefficient $\alpha_n \neq 0$, since each non-zero coefficient α_n for $n > 0$ yields two linearly independent solutions (namely, $\sin 2n\pi p^{-1}t$, $\cos 2n\pi p^{-1}t$) and since p is assumed minimal,we see that (H)has exactly two independent p-periodic solutions $\sin 2\pi p^{-1}t$ and $\cos 2\pi p^{-1}t$ if and only if,

$$\begin{aligned} &\lambda_1 + \lambda_2 \neq 0 && (n = 0) \\ &\lambda_1 + \lambda_2 C_1(p) = 0, \; \lambda_2 S_1(p) = 2\pi p^{-1} && (n = 1) \\ &\lambda_1 + \lambda_2 C_n(p) \neq 0, \; \lambda_2 S_n(p) \neq 2\pi np^{-1} && (n \geq 2) \end{aligned}$$

We also have

$$|(\lambda_1+\lambda_2 C_n(p)-i(2\pi np^{-1}-\lambda_2 S_n(p)))| > 0$$

and so the corresponding α_ns are zero and the number of linearly independent solutions are the same.

(b) If $f(t) = \sum_{m=-\infty}^{\infty} f_m e^{2\pi imp^{-1}t}$ is substituted into the nonhomogeneous equation (NH), we obtain,

$$(\lambda_1+\lambda_2 C_n(p)+i(2n\pi p^{-1}-\lambda_2 S_n(p)))\alpha_n = -f_n; \quad -\infty < n < +\infty \qquad (3.18)$$

which is to be solved for α_n. If $k = 0$ i.e. the homogeneous equation possesses only the trivial solution $y \equiv 0$, then by the proof of (a) above, each coefficient is invertible and hence (3.18) may be uniquely solved for α_n. Thus the Fourier series (3.15) will define a real valued solution $y \in P(p)$ for these unique α_n provided we can show that the series defines an absolut ly continuous function.

It is clear, since $C_n(p)$ and $S_n(p)$ are bounded independently of n, that

$$|\alpha_0| \leq k f_0 \quad \text{and } n\,|\alpha_n| \leq k\,|f_n| \quad \text{for } n \neq 0$$

for some constant k independently of n. Thus

$$\sum_{n=-\infty}^{\infty} |\alpha_n|^2 n^2 \leq k^2 \sum_{n=-\infty}^{\infty} |f_n|^2 = k^2 ||f||_2^2 < +\infty .$$

where $||f||_2$ is the L^2 norm of $f \in P(p)$ on $[0,p]$. The Riesz-Fischer theorem implies that $y(t)$ defined by (3.15) lies in L^2 while the above estimate shows that $y(t)$ is absolutely continuous.

(c) Let $N = \{ n : |\lambda_1+\lambda_2 C_n(p)+i(2\pi np^{-1}-\lambda_2 S_n(p))| = 0 \}$

This part follows straight forwardly from the fact that (3.18) has a solution α_n for $n \in N$ if and only if $f_n \alpha_n = 0$ for all solution α_n, $n \in N$ of the adjoint system (3.17) and that in this case, there is a unique solution α_n orthogonal to the solution space of (3.16) which satisfies $n|\alpha_n| \leq k|f_n|$ $n \in N$. We deduce the absolute continuity in the same way as in (b).

This proves the lemma.

We are interested in those values of λ_i for which the homogeneous equation (H) and therefore the adjoint equation (A) possess exactly two linearly independent non-trivial p-periodic solutions.

Coming back to equation (3.16)

$$(\lambda_1+\lambda_2 C_n(p) + i(2\pi np^{-1}-\lambda_2 S_n(p)))\alpha_n = 0 \; ; \; n \geq 0.$$

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$$\begin{aligned}
&\lambda_1 + \lambda_2 \neq 0 && (n = 0) \\
&\lambda_1 + \lambda_2 C_1(p) = 0, \; \lambda_2 S_1(p) = 2\pi p^{-1} && (n = 1) \\
&\lambda_1 + \lambda_2 C_n(p) \neq 0, \; \lambda_2 S_n(p) \neq 2\pi np^{-1} && (n \geq 2)
\end{aligned}$$

The second pair of equations yield the values of λ_1 and λ_2 in which we are interested. In fact $\lambda_1 = -2\pi C_1(p)/pS_1(p)$, $\lambda_2 = 2\pi/pS_1(p)$. For these values of λ_1 and λ_2, $\sin 2\pi p^{-1}t$ and $\cos 2\pi p^{-1}t$ are also solution to the adjoint equations. We are now in a position, thanks to Fredholm's alternative, to define projection operators P and Q.

Let $X = Y = L^\infty[0,p]$. Let us think of $L^\infty[0,p]$ as contained in $L_2[0,p]$.

$$\text{Let } R(P) = P(X) = X_o, \ \operatorname{Ker}P = Q(I-P)X = X_1$$

$$R(Q) = Q(Y) = Y_o, \ \operatorname{Ker}Q = R(I-Q) = (I-Q)Y = Y_1.$$

Let $w_1 = \emptyset_1 = \sin 2\pi p^{-1}t$, $w_2 = \emptyset_2 = \cos 2\pi p^{-1}t$.

We define projection operators.

$$Px = \sum_{i=1}^{2} c_i\emptyset_i = \sum_{i=1}^{2} (x,\emptyset_i)\emptyset_i, \ (x,\emptyset_i) = \int_o^p x(t)\emptyset_i(t)dt.$$

$$Qy = \sum_{i=1}^{2} d_i w_i = \sum_{i=1}^{2} (y,w_i)w_i, \ (y,w_i) = \int_o^p y(t)w_i(t)dt.$$

Then P and Q are the same orthogonal projections and $\dim X_o = \dim Y_o = 2$.

In what follows we designate by L and L^* respectively, the linear operators

$$Lx = x'(t) + \lambda_1 x(t) + \lambda_2 \int_o^\infty x(t-s)dh(s).$$

and $$L^* z = z'(t) - \lambda_1 z(t) - \lambda_2 \int_o^\infty dh(s)z(t+s).$$

$L|_{D(L)\cap N(L)^\perp} \to R(L)$ is a 1-1 closed linear operator (being the sum of a closed linear operator and a bounded linear

operator defined on the whole space) and having the same range as L where $L|_{D(L)\cap N(L)^{\perp}}$ denotes the restriction of L to $D(L)\cap N(L)^{\perp}$. Let H denote the inverse of the operator $H = [L|_{D(L)\cap N(L)^{\perp}}]^{-1}$. By the closed graph theorem, H is a one-one continuous linear operator

$$D(H) = R(L) \qquad R(H) = D(L)\cap N(L)^{\perp}$$

Moreover,

$$LHy = y \text{ for all } y \in R(L)$$

$$HLx = x - \sum_{i=1}^{2} (x,\phi_i)\phi_i \quad \forall \; x \in D(L).$$

The operators P and Q have the following properties

(i) P and Q are continuous linear operators defined on the whole of $L^{\infty}[0,p] \subseteq L^2[0,p]$

(ii) $R(P) = \langle \phi_1,\phi_2 \rangle$, $R(Q) = \langle w_1,w_2 \rangle$.

(iii) $P^2 = P$ and $Q^2 = Q$.

(iv) The range of (I-Q) is a subset of R(L) and H(I-Q) is a continuous linear operator defined on all of $L^{\infty}[0,p]$.

The proof of the first three statements are easy to see and the fourth follows from the fact $R(Q) = N(L^*)$.
Also H(I-Q) is continuous linear operator because I-Q is bounded and H is continuous by the closed graph theorem. We now have the following.

<u>Lemma 2</u>.

(i) $H(I-Q)Lx = (I-P)x \quad \forall \; x \in D(L)$.

(ii) $LH(I-Q)x = (I-Q)x \qquad \forall\, x \varepsilon\, X = Y = L^{\infty}[0,p].$

(iii) $LPx = QLx \qquad \forall\, x \varepsilon\, D(L)$

<u>Proof</u>. (i) Let $x \varepsilon D(L)$

$$(I-Q)Lx = Lx - \sum_{i=1}^{2} (Lx, w_i)w_i$$

$$= Lx - \sum_{i=1}^{2} (x, L^* w_i)w_i$$

$$= Lx \text{ since } L^* w_i = 0,\ w_i \varepsilon N(L^*),\ i = 1,2.$$

Hence $H(I-Q)Lx = HLx$

$$= x - \sum_{i=1}^{2} (x, \phi_i)\phi_i$$

$$= (I-P)x$$

(ii) Since $(I-Q)x \varepsilon R(L)$ for all $x \varepsilon L^{\infty}[0,p]$ it follows that $LH(I-Q)x = (I-Q)x$.

(iii) Let $x \varepsilon D(L)$. Then

$$Px = \sum_{i=1}^{2} (x, \phi_i)\phi_i$$

Therefore $LPx = \sum_{i=1}^{2} (x, \phi_i)L\phi_i = 0$ since $(\phi_1, \phi_2) \varepsilon N(L)$.

Also $QLx = \sum_{i=1}^{2} (Lx, w_i)w_i$.

$$= \sum_{i=1}^{2} (x, L^* w_i)w_i = 0 \text{ since } w_i \varepsilon N(L^*).$$

Hence $LPx = QLx$.

This proves the lemma.

Let now M, k_o, k_o', χ_o, χ_o' be the norms of H, P, (I-P), Q and (I-Q) in the uniform topologies of X and Y. For every $x^* \varepsilon X_o$, we have $x^* = \sum_{i=1}^{2} c_i w_i$ or briefly $x^* = cw, c = \langle c_1, c_2 \rangle \varepsilon R^2$ and there are constants $0 < \gamma' \leq \gamma < \infty$ such that $\gamma' |c| \leq ||cw|| \leq \gamma |c|$ for all $x^* = cw \varepsilon X_o$ and where $|.|$ denotes the Euclidean norm in R^2. Let $\gamma_o = \min[1, \gamma]$. Also there is a constant $\mu > 0$ such that for every $y \varepsilon Y$ and $d = \langle y, w \rangle$ or $d = \langle d_1, d_2 \rangle \varepsilon R^2$, $d_i = \langle y, w_i \rangle$, $i = 1,2$ we have $|d| \leq \mu ||y||$. For $x^* \varepsilon X_o$, $x^*(t) = c_1 w_1 + c_2 w_2$ with $c = \langle c_1, c_2 \rangle$ we take $b = c/|c|$ or $b = \langle b_1, b_2 \rangle$, $b_1^2 + b_2^2 = 1$, $b_i = c_i/|c|$ $i = 1,2$ and $v(t) = b_1 w_1 + b_2 w_2$, $x^*(t) = |c| v(t)$, $t \varepsilon [0, p]$.

For any number $\sigma \geq 0$ let $D_b(\sigma)$ denote the set of all points $t \varepsilon G$ where $|v(t)| \leq \sigma$. It is easy to see that for some constant $\rho > 0$, $p^{-1} \int_0^p |v(t)|\, dt \geq \rho$ for all functions $v(t) = b_1 w_1 + b_2 w_2$ with $b_1^2 + b_2^2 = 1$. Indeed, since $w_1 = \sin 2\pi p^{-1} t$ and $w_2 = \cos 2\pi p^{-1} t$ and $b_1^2 + b_2^2 = 1$, taking $b_1 = \cos\theta$, $b_2 = \sin\theta$, our integral becomes

$$p^{-1} \int_0^p |\sin(2\pi p^{-1} t + \theta)|\, dt = \frac{2}{\pi} > 0.$$

Let η_1, ε be positive constants with $\rho\eta_1 > \varepsilon \gamma_o$ and take $Nx = g(x, \lambda_1, \lambda_2)$, $x \varepsilon X = L^\infty[0,p]$.

For the function $g(s, \lambda_1, \lambda_2)$ we consider the monotonic nondecreasing function $\phi(\xi)$, $0 \leq \xi < +\infty$, defined by

$$\phi(\xi) = \sup[|g(s, \lambda_1, \lambda_2)|, |s| \leq \xi, \lambda_1, \lambda_2 \varepsilon R].$$

Furthermore we assume that

(a) there is a constant $R_o > 0$ such that $g(s,\lambda_1,\lambda_2) \leq -\eta_1$ for $s \geq R_o$ and $g(s,\lambda_1,\lambda_2) \geq \eta_1$ for $s \leq -R_o$.

Let K be a positive constant. We have the following lemma.

Lemma 3. For $P = Q$ the same orthogonal projection, under hypothesis on $g(x,\lambda_1,\lambda_2)$ and ρ, and for $\rho\eta_1 \geq \varepsilon\gamma_o$, there is a constant $R_1 \geq R_o+K$ such that $\langle QNx, x^*\rangle \leq -\varepsilon\|x^*\|_\infty$ for all $x \in X$, $x^* \in X_o$ with $Px = x^*$, $\|x^*\| \geq R_1$, $\|x-x^*\| \leq K$.

Proof. For $Nx = g(x,\lambda_1,\lambda_2)$.

$$QNx = \sum_{i=1}^{2} w_i(t)\int_0^p g(x(\alpha),\lambda_1,\lambda_2)w_i(\alpha)d\alpha .$$

$$\langle QNx, x^*\rangle = p^{-1}\sum_{i=1}^{2} c_i\int_0^p g(x(\alpha),\lambda_1,\lambda_2)w_i(\alpha)d\alpha .$$

$$= p^{-1}\int_0^p g(x,\lambda_1,\lambda_2)x^*(t)dt.$$

Let $\lambda > 0$ be a constant such that $\rho\eta_1 - 2\lambda \geq \varepsilon\gamma_o$. Let R_1 be any number such that

$$R_1 \geq \max[\lambda^{-1}\gamma_o(R_o+K)\phi(R_o+2K), \lambda^{-1}\gamma_o\eta_1(R_o+K)].$$

Let us assume $\|x^*\|_\infty \geq R_1$ and $\|x-x^*\|_\infty \leq K$. Let $\sigma = (R_o+K)|c|^{-1}$. We now have

$$x^* = |c|v,\ \gamma_o'|c| \leq \|x^*\|_\infty = \|cw\|_\infty \leq \gamma_o|c|.$$

$$\langle QNx, x^*\rangle = |c|\left[p^{-1}\int_{D_b(\sigma)} g(x(t),\lambda_1,\lambda_2)v(t)dt + \int_{[0,p]-D_b(\sigma)} g(x(t),\lambda_1,\lambda_2)v(t)dt\right].$$

For $t \in D_b(\sigma)$ we have $|v(t)| \leq \sigma = (R_o+K)|c|^{-1}$,

hence $|x^*(t)| = |c||v(t)| \leq R_o+K$. Hence

$$|x(t)| = |x^*(t) + (x(t) - x^*(t))|$$

$$\leq |x^*(t)| + ||x-x^*||_\infty \leq R_o+K+K = R_o+2K$$

and $$|g(x,\lambda_1,\lambda_2)| \leq \emptyset(R_o+2K).$$ Hence

$$p^{-1} \int_{D_b(\sigma)} g(x,\lambda_1,\lambda_2)v(t)dt \leq p^{-1} |D_b(\sigma)| \emptyset(R_o+2K)\sigma$$

$$\leq \emptyset(R_o+2K)(R_o+K) |c|^{-1}$$

$$\leq \emptyset(R_o+2K)(R_o+K) v_o R_1^{-1} \leq \lambda.$$

Thus for $t \in [0,p]-D_b(\sigma)$, either $x^*(t) \geq R_o+K$, and $x(t) = x(t)-(x^*(t)-x(t)) \geq x^*(t)-||x-x^*||_\infty$

$$\geq R_o+K-K = R_o$$

and then $g(x,\lambda_1,\lambda_2) \leq -\eta_1$; or $x^*(t) \leq -R_o-K$, and

$$x(t) = x^*(t)-(x^*(t)-x(t))$$

$$\leq x^*(t) + ||x-x^*||_\infty \leq -R_o-K+K = -R_o$$

and then $g(x,\lambda_1,\lambda_2) \geq \eta_1$. In any case, $g(x,\lambda_1,\lambda_2)v(t) \leq -\eta_1|v(t)|$ for all $t \in [0,p]-D_b(\sigma)$. Thus,

$$p^{-1} \int_{[0,p]-D_b(\sigma)} g(x,\lambda_1,\lambda_2)v(t)dt$$

$$\leq -p^{-1} \int_{[0,p]-D_b(\sigma)} \eta_1|v(t)|dt$$

$$= -p^{-1} \int_0^p \eta_1|v(t)|dt + p^{-1} \int_{D_b(\sigma)} \eta_1|v(t)|dt$$

$$\leq \eta_1\rho + p^{-1}|D_b(\sigma)|\eta_1 \leq -\eta_1\rho + \eta_1(R_o+K)|c|^{-1}$$

$$\leq -\eta_1\rho + \eta_1(R_o+K)|c|^{-1}$$

$$\leq \; -\eta_1\rho + \eta_1(R_o+K)\,\gamma_o R_1^{-1}$$

$$\leq \; \lambda - \rho\eta_1$$

Finally, $\langle QNx, x^* \rangle \leq |c| \, [\lambda+\lambda-\rho\eta_1] = - \, |c| \, [\rho\eta_1 - 2\lambda]$

$$\leq \; -\gamma_o^{-1}(\rho\eta_1 - 2\lambda)\|x^*\|_\infty$$

$$\leq \; -\varepsilon\|x^*\|_\infty \, .$$

This proves the lemma.

If we define $J : R(Q) \rightarrow R(P)$ as the identity operator, then the requirement $\langle JQNx, x^* \rangle \leq 0$ becomes $\langle QNx, x^* \rangle \leq 0$. In view of the lemma above, which proves something stronger, we are assured of the existence of a solution to (N) under the stipulated growth condition on the functional $g(x, \lambda_1, \lambda_2)(t)$.

$$\tilde{x}(t-1) = \begin{cases} x(t-1) \text{ if } (t-1) \varepsilon [0,2\pi] \\ x(2\pi+t-1) \text{ if } t-1 < 0 \end{cases}$$

We have $f(t,x(t),\tilde{x}(t-1)) = - |x|^{1/2}\arctan x(t) - k_o \arctan x(t-1)$. Choose now a constant $R_o > 0$ and denote by the set $S \subsetneq C[0,2\pi]$ all functions $x(t)$ such that $|x(t)-x(t+\theta)| < \frac{R_o}{2}$ whenever $|\theta| \leq 1$. In this set S for suitable $R_o, x(t) \geq R_o \Rightarrow x(t-1) > 0$ and $x(t) \leq -R_o \Rightarrow x(t-1) < 0$. Then

$$f(t,x(t),\tilde{x}(t-1)) \begin{cases} \leq 0 \text{ for } x(t) \geq R_o. \\ \geq 0 \text{ for } x(t) \leq -R_o. \end{cases}$$

Also if we define $\phi(\xi) = \sup[\,|h(t)+f(t,s_1,s_2)|\;\; t \varepsilon [0,2\pi]$, $\max(|s_1|,|s_2|) \leq \xi]$, then $\frac{\phi(\xi)}{\xi} \to 0$ as $\xi \to \infty$.

Consider now the functional differential equation

$$x'' + \alpha x^2 \sin t + \arctan x(t) + k_o \arctan x(t-1) = \sin t + \eta \sin^2 t.$$

where α is a contant and k_o a small positive constant. This can be formulated in the more abstract form $Lx + \alpha Ax = Nx$ where L is as above, the differential operator x'' with the boundary conditions $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$. $A : C[0,2\pi] \to C[0,2\pi]$ is a continuous operator given by $(Ax)(t) = g(t,x(t))$, $g(t,x) = x^2 \sin t$. $(Nx)(t) = h(t)+f(t,x(t),\tilde{x}^2(t-1))$ where $h(t) = \sin t + \eta \sin^2 t$ and $f(t,x(t),\tilde{x}(t-1)) = -\arctan x(t) - k_o \arctan x(t-1)$. Here $g(t,x)$ is a continuous function and $h(t)$ satisfies $\frac{1}{2\pi}\int_0^{2\pi} h(t)dt < \eta$. In our set S, we can see that there exists an η_1 and an R_o such that $f(t,x(t),\tilde{x}(t-1)) \leq -\eta_1$ for $x(t) \geq R_o$ and $f(t,x(t),\tilde{x}(t-1)) > \eta_1$

for $x(t) \leq -R_o$. Also if we define $\emptyset(\xi) = \sup[|h(t)+f(t,s_1,s_2)|;$ $t \varepsilon [0,2\pi]$, $\max(|s_1|, |s_2|) \leq \xi]$, then $\frac{\emptyset(\xi)}{\xi} \to 0$ as $\xi \to \infty$.

As in the previous chapter we can convert the problem of establishing a solution of the abstract operator equation in S into an equivalent fixed point problem for the operator T given by

$$Tx = Px + H(I-Q)Nx + JQNx \tag{4.1}$$

where $J \circ R(Q) \to R(P)$ is an isomorphism. Assume for the time being that the operator T is compact. We prove this fact in a lemma at the end of the chapter.

Let $[a,b]$ be a closed and bounded interval on the real line. $X = Y = C[a,b]$. Let S denote the open set of all functions belonging to $C[a,b]$ such that $|x(t)-x(t+\theta)| < \frac{R_o}{2}$ whenever $|\theta| \leq \tau < b-a$. Let $<y,x>$ denote the linear operation $(b-a)^{-1}\int_a^b y(t)x(t)dt$ defined for all $x \varepsilon X$ and $y \varepsilon Y$ and satisfying $|<y,x>| \leq ||x||_\infty ||y||_\infty$, where $||.||_\infty$ denotes the norm in L $[a,b]$.

Let $h(t)$, $t \varepsilon [a,b]$ be any element of $Y = C[a,b]$ and let $f(t,x,y)$ be any continuous function defined on $[a,b]$xSxS. Let Nx be the operator defined by

$$(Nx)(t) = h(t) + f(t,x(t),x(t-\tau)) \tag{4.2}$$

Here we note that $x(t-\tau)$ may not be defined for all $t \varepsilon [a,b]$ where τ is a positive constant less than b-a. Hence we extend the function periodically by

$$\tilde{x}(t-\tau) = \begin{cases} x(t-\tau) & \text{if } t-\tau \in [a,b] \\ x(b-a+t-\tau) & \text{if } t-\tau < a. \end{cases}$$

It is clear that $N:S \to Y$.

Let P and Q denote projection operators on X and Y defined by

$$Px = (b-a)^{-1} \int_a^b x(t)dt \qquad Qy = (b-a)^{-1} \int_a^b y(t)dt \quad (4.3)$$

Let $R(P) = R(Q)$ and $J:R(Q) \to R(P)$ be the identity operator. Then the condition $\langle JQNx, x^* \rangle \leq 0$ (or ≥ 0) becomes $\langle QNx, x^* \rangle \leq 0$ (or ≥ 0). Let $||H|| = M$, $k_0 = ||P||$, $k' = ||I-P||$, $\gamma = ||Q||$, $\gamma' = ||I-Q||$. We shall assume that

$$(H_0) \int_0^b h(t)dt = 0$$

$$(F_0) \quad f(t,x(t),\tilde{x}(t-\tau)) \begin{cases} \leq 0 & \text{for } x(t) \geq R_0 \\ \geq 0 & \text{for } x(t) \leq -R_0. \end{cases}$$

Let $k \geq 0$ be any constant and choose $\bar{R} \geq R_0+k$.

<u>Lemma 1</u>. For P,Q defined by (4.3) and under hypotheses (H_0), (F_0) we have $\langle QNx, x^* \rangle \leq 0$ for all $x \in X$, $x^* \in X_0$ with $Px = x^*$, $||x^*||_\infty = \bar{R}$, $||x-x^*||_\infty \leq K$.

<u>Proof</u>. Indeed, $x^* = Px = (b-a)^{-1} \int_a^b x(t)dt = c$

$$QNx = (b-a)^{-1} \int_a^b [h(t)+f(t,x(t),\tilde{x}(t-\tau)]dt = d .$$

c,d are real numbers and hence $\langle QNx, x^* \rangle = cd$.

For $||x^*||_\infty = \bar{R}$, i.e. $|c| = \bar{R}$ we have either $c = \bar{R}$ and then $x(t) = c-(c-x(t)) \geq c-||x-x^*||_\infty \geq \bar{R}-k \geq R_0$ or $c = -\bar{R}$

and then $x(t) = c-(c-x(t)) \leq c+\|x-x^*\|_\infty \leq -\bar{R}+k \leq -R_o$.
Correspondingly, in our set S we have $f(t,x(t),\tilde{x}(t-\tau)) \leq 0$, $d \leq 0$ in the first case or $f(t,x(t),\tilde{x}(t-\tau)) \geq 0$, $d \geq 0$ in the second case and again $cd \leq 0$. Q.E.D.

Let η_1,η_2 be positive constants with $\eta_1-\eta_2 \geq \varepsilon$. Let us assume that

$$(H) \quad \int_a^b h(t)dt \leq \eta_2$$

$$(F_\eta) \quad f(t,x(t),x(t-\tau)) \quad \begin{cases} \geq \eta_1 & \text{for } x(t) \leq -R_o. \\ \leq -\eta_1 & \text{for } x(t) \geq R_o. \end{cases}$$

Let $k \geq 0$ be any constant and take $\bar{R}_1 \geq R_o+k$.

<u>Lemma 2</u>: For P and Q defined by (4.3), under hypothesis (F_η), (H) and if $\eta_1-\eta_2 \geq \varepsilon$, then $\langle QNx, x^* \rangle \leq -\varepsilon\|x^*\|_\infty$ for all $x \in X$, $x^* \in X_o$ with $Px = x^*$, $\|x^*\|_\infty = \bar{R}$, $\|x-x^*\|_\infty \leq k$.

<u>Proof</u>. As before $x^* = Px = (b-a)^{-1}\int_a^b x(t)dt = c$;
$QNx = (b-a)^{-1}\int_a^b [h(t)+f(t,x(t),\tilde{x}(t-\tau))] = d_o$, $\langle QNx, x^* \rangle = cd$ and for $c = \bar{R} \geq R_o+k$, we have $x(t) \geq R_o$ and $f(t,x(t),\tilde{x}(t-\tau)$ $\leq -\eta_1$, $d \leq -\eta_1+\eta_2 \leq -\varepsilon c$; $cd \leq -\varepsilon|c| = -\varepsilon\|x^*\|_\infty$.

Analogously, for $c = -\bar{R} \leq -R_o-k$, we have $x(t) \leq -R_o$, $f(t,x(t),\tilde{x}(t-\tau) \geq \eta_1$, $d \geq \eta_1-\eta_2 \geq \varepsilon$, $cd \leq -\varepsilon|c| = -\varepsilon\|x^*\|_\infty$.

Again let S $C[a,b]$, $Y = C[a,b]$, $S_o = PS$, $Y_o = QY$ with $1 \leq m = \dim S_o < +\infty$. Here we note that since S is merely an open set, S_o need not be a linear manifold. Let us think of S and Y as contained in $L^2[a,b]$ and let us assume that there exists a basis $w = (w_1,\ldots,w_m)$ for S_o and Y_o made up

of elements that are orthonormal in $L^2[a,b]$. For P,Q the same orthogonal projections, then

$$Px = \sum_{i=1}^{m} c_i w_i \qquad c_i = \int_a^b x(t)w_i(t)dt \quad i = 1,..m \quad x \in S$$

$$(4.4)$$

$$Qy = \sum_{i=1}^{m} d_i w_i \qquad di = \int_a^b y(t)w_i(t)dt \quad i = 1,..m \quad y \in Y.$$

For $x^* \in X_o$, $x^*(t) = c_1w_1 + \ldots c_m w_m$ with $c = (c_1,\ldots,c_m) \neq 0$, take $\ell = \frac{c}{|c|}$ or $\ell = (\ell_1,\ldots,\ell_m)$, $|\ell| = 1$, $\ell_i = \frac{c_i}{|c|}$, $v(t) = \ell_1 w_1 + \ldots + \ell_m w_m$, $x^*(t) = |c|\, v(t)$, γ_o and γ'_o are two constants such that $\gamma'_o\, |c| \leq \ ||x^*||_\infty = ||cw|| \leq \gamma_o\, |c|$.

For any number σ, let

$$D(\sigma) = \{t \in [a,b] \,;\, |v(t)| \leq \sigma\} \tag{4.5}$$

We assume the following

(p^*) There is a constant $\rho > 0$ such that $(b-a)^{-1} \int_a^b |v(t)|\, dt \geq \rho$ for all functions $v(t) = \ell_1 w_1 + \ldots + \ell_m w_m$ with $|\ell| = 1$.

This condition is usually satisfied in applications with smooth functions.

(H^*) $(b-a)^{-1} \int_a^b h(t)v(t)dt| \leq \eta_2$ for all $v(t) = \ell_1 w_1 + \ldots + \ell_m w_m$ with $|\ell| = 1$ where $\ell = (\ell_1,\ldots,\ell_m)$.

For the function $f(t,s_1,s_2)$ continuous on $[a,b]xSxS$ we consider the monotonic non-decreasing function $\phi(\xi)$, $0 \leq \xi < +\infty$ defined by

$\phi(\xi) = \sup[\,|f(t,s_1,s_2)| : \max(|s_1|\,,\,|s_2|) \leq \xi]$ such that $\frac{\phi(\xi)}{\xi} \to 0$ as $\xi \to +\infty$. Further we assume that

(F^*) There is a constant $R_o > 0$ such that

$$f(t,s_1,s_2) \begin{cases} < -\eta_1 & \text{for } s_1 \geq R_o \\ > \eta_1 & \text{for } s_1 \leq -R_o. \end{cases}$$

<u>Lemma 3</u> : For $P = Q$ the orthogonal projections defined by (4.4) under hypothesis (p^*), (H^*), (F^*) and for $\rho\eta_1 - \eta_2 > \varepsilon\gamma_o$, there exists a constant $\bar{R} \geq R_o + k$ such that $< QNx, x^* > \leq -\varepsilon \| x^* \|_\infty$ for all $x \in S$, $x^* \in S_o$ with $Px = x^*$, $\| x^* \|_\infty = \bar{R}$, $\| x - x^* \|_\infty \leq k$.

<u>Proof</u>. For $Nx = h(t) + f(t, x(t), \check{x}(t-\tau))$,

$$QNx = \sum_{i=1}^{m} w_i(t) \int_a^b [h(\alpha) + f(\alpha, x(\alpha), \check{x}(\alpha - \tau)]d\alpha$$

$$< QNx, x^* > = (b-a)^{-1} \sum_{i=1}^{m} c_i \int_a^b [h(\alpha) + f(\alpha, x(\alpha), \check{x}(\alpha-\tau)]d\alpha$$

$$= (b-a)^{-1} \int_a^b [h(t) + f(t, x(t), \check{x}(t-\tau))]x^*(t)dt$$

Let $\lambda > 0$ be a constant such that $\rho\eta_1 - \eta_2 - 2\lambda \geq \varepsilon\gamma_o$. Let $\bar{R}$ be any number such that

$$\bar{R} \geq \max[\lambda^{-1}\gamma_o(R_o+k)\varnothing(2\bar{R}+k), \lambda^{-1}\gamma_o\eta_1(R_o+k)].$$

Let us assume $\| x^* \|_\infty = \bar{R}$ and $\| x - x^* \|_\infty \leq K$. Let $\sigma = (R_o+k)|c|^{-1}$

We now have, $x^* = |c|v$, $\gamma_o' |c| \leq \| x^* \|_\infty = \| cw \|_\infty \leq \gamma_o |c|$

$$< QNx, x^* > = |c| [(b-a)^{-1} \int_a^b h(t)v(t)dt$$

$$+ (b-a)^{-1} \int_{D(\sigma)} f(t, x(t), x(t-\tau))v(t)dt$$

$$+ (b-a)^{-1} \int_{[a,b]-D(\sigma)} f(t, x(t), x(t-\tau)v(t)dt$$

For $t \in D(\sigma)$, $|v(t)| \leq \sigma = (R_o+k)|c|^{-1}$, $|x^*(t)| = |c||v(t)| \leq R_o+k$.

Hence $|x(t)| = |x^*(t)+x(t)-x^*(t)| \leq |x^*(t)| + ||x-x^*|| \leq$ $R_o+k+k = R_c+ 2k \leq \bar{R}+k$.

Similarly, $|x(t-\tau)| \leq 2\bar{R}+k$ and $|f(t,x(t),\tilde{x}(t-\tau))| \leq \phi(2\bar{R}+k)$.

Hence $(b-a)^{-1} \int_{D(\sigma)} f(t,x(t),\tilde{x}(t-\tau)v(t)dt \leq (b-a)^{-1}|D(\sigma)|\phi(2\bar{R}+k)\sigma$

$\leq \phi(2\bar{R}+k)(R_o+k)\ |c|^{-1} \leq \phi(2\bar{R}+k)(R_o+k)\gamma_o(\bar{R})^{-1} \leq \lambda.$

For $t \in [a,b]-D(\sigma)$, either $x^*(t) \geq R_o+k$ and

$$x(t) = x^*(t)-(x^*(t)-x(t)) \geq x^*(t)-||x-x^*||_\infty \geq R_o+k-k = R_o$$

and then $f(t,x(t),\tilde{x}(t-\tau)) \leq -\eta_1$

or $x^*(t) \leq -R_o-k$ and $x(t) = x^*(t)-(x^*(t)-x(t)) \leq x^*(t)+||x-x^*||_\infty$ $\leq -R_o-k+k = R_o$ and then $f(t,x(t),x(t-\tau)) \geq \eta_1$.

In any case we have $f(t,x(t),x(t-\tau))v(t) \leq -\eta_1|v(t)|$ for all $t \in [a,b]-D(\sigma)$. Thus $(b-a)^{-1} \int_{[a,b]-D(\sigma)} f(t,x(t),\tilde{x}(t-\tau))v(t)dt$

$$\leq -(b-a)^{-1} \int_{[a,b]-D(\ )} \eta_1 |v(t)|dt =$$

$$= -(b-a)^{-1}\int_a^b \eta_1|v(t)|dt + (b-a)^{-1} \int_{D(\sigma)} \eta_1|v(t)|\,dt$$

$$\leq -\eta_1\rho + (b-a)^{-1}|D(\sigma)|\,\eta_1\sigma \leq -\eta_1\rho+\eta_1(R_o+k)\ |c|^{-1}$$

$$\leq -\eta_1\rho+\eta_1(R_o+k)\gamma_o(\bar{R})^{-1} \leq \lambda-\rho\eta_1.$$

Finally $\langle QNx,x^*\rangle \leq |c|\,[\eta_2+\lambda+\lambda-\rho\eta_1] = -|c|\,[\rho\eta_1-\eta_2-2\lambda]$

$$\leq -\gamma_o^{-1}(\rho\eta_1-\eta_2-2\lambda)\,||x^*||_\infty \leq -\varepsilon||x^*||_\infty.$$

The same results hold good if we define

$D(\sigma) = \{t \in [a,b]: |v(t+\tau)| \leq \sigma\}$ and replace condition

(F^*) by

(F^{**})

$$f(t,s_1,s_2)\begin{cases}\leq & -\eta_1 \text{ for } s_2 \geq R_o\\ \geq & \eta_1 \text{ for } s_2 \leq -R_o.\end{cases}$$

Only in this case we replace the integral

$$(b-a)^{-1}\int_{D(\sigma)} f(t,x(t),x(t-\tau))v(t)dt$$

by $$(b-a)^{-1}\int_{D(\sigma)-\tau} f(\theta+\tau,\ x(\theta+\tau),\ x(\theta))v(\theta+\tau)d\theta\ .$$

We formulate the following theorem.

Theorem 1. Let S denote the set of all functions $x(t)\ \varepsilon\ C[a,b]$ such that $|x(t)-x(t+\theta)| < \frac{R_o}{2}$ whenever $|\theta| \leq \tau < b-a$. $X = Y = C[a,b]$. Let P,Q be the operators defined by (4.3). Let L,H be the linear operators satisfying the properties (i), (ii),(iii) of section 2 of chapter 3 with $X_o = PX = \text{Ker } L$ nontrivial and finite dimensional and H linear, bounded and compact and let $f(t,s_1,s_2)$ be a real valued continuous function on $[a,b]xSxS$. Let h(t) be an element of $C[a,b]$ with $\int_a^b h(t)dt=0$. Assume that there exists a constant $R_o > 0$ such that $f(t,s_1,s_2) \leq 0$ for $s_1 \geq R_o$ and $f(t,s_1,s_2) \geq 0$ for $s_1 \leq -R_o$. Assume that $\sup_{t\varepsilon[a,b]} |h(t)+f(t,x(t),\tilde{x}(t-\tau))| \leq J_o$, $M\,\varkappa' J_o < R_o/4$ and assume further that a basis $(w_1,\ldots,w_m)$ exists for S_o. Then the equation $(Lx)(t) = h(t)+f(t,x(t),\tilde{x}(t-\tau)), \theta < \tau < b-a$, $t\ \varepsilon\ [a,b]$ has at least a solution $x\ \varepsilon\ D(L) \subseteq C[a,b]$.

Lemma : Let $||Nx|| \leq J_o$ and $M\,\varkappa' J_o < R_o/4$. Then for $||Px|| < R_o$ and $||x|| < c$, there is no solution of $x=Px+H(I-Q)Nx+JQNx$ for which $|x(t)-x(t+\theta)| = \frac{R_o}{2}$ for $|\theta| \leq \tau < b-a$.

<u>Proof.</u> Let $x(t) = (Px)(t)+(H(I-Q)Nx)(t)+(JQNx)(t)$

$$x(t+\theta) = (Px)(t+\theta)+(H(I-Q)Nx)(t+\theta)+(JQNx)(t+\theta)$$

Hence

$$|x(t)-x(t+\theta)| = |(H(I-Q)Nx)(t)-(H(I-Q)Nx)(t+\theta)|$$

$$\leq ||H|| \; ||I-Q|| \; (|Nx)(t)| + |(Nx)(t+\theta)|)$$

$$\leq \; 2M \, \alpha' J_o$$

or $R_o/2 < R_o/2$ which is a contradiction. Hence the lemma follows. Q.E.D.

<u>Proof of Theorem 1.</u> Theorem 1 now follows from the above mentioned lemma and Theorem 1 of Chapter 3.

To complete the result however we have to show that the operator T is compact. This we do in the form of a lemma below.

<u>Lemma.</u> The operator T defined by $Tx = Px+H(I-Q)Nx+JQNx$ is compact.

<u>Proof.</u> P and Q are finite dimensional and J is a one-one isomorphism from $R(P) \to R(Q)$. Hence P and JQN are compact. Hence it is sufficient to show that H is compact. Let $\{y_n\}$ be a bounded sequence and $x_n = Hy_n$; $||y_n||_\infty \leq M_1 \;\; \forall\, n$. $H.R(L) \to D(L) \cap (\mathrm{Ker}L)^{\perp}$. Since $x_n \to D(L)$, they are once continuously differentiable and bounded because H as a linear operator is bounded

$$x = Hy = \int_a^b k(t,s)y(s)ds$$

$$x' = \int_a^b \frac{\partial k(t,s)}{\partial t} y(s)ds$$

$$
\begin{aligned}
|x'| &= \left|\int_a^b \frac{\partial k(t,s)}{\partial t}\, y(s)ds\right| \\
&\leq \int_a^b \left|\frac{\partial k(t,s)}{\partial t}\, y(s)\right|ds \\
&\leq \left(\int_a^b \left(\frac{\partial k(t,s)}{\partial t}\right)^2 ds\right)^{1/2} \left(\int_a^b |y(s)|^2 ds\right)^{1/2} \\
&\leq M_2 M_3 (b-a)
\end{aligned}
$$

Now by the uniform boundedness for x_n and the uniform boundedness for x_n', that is equicontinuity for x_n, we have by Arzela Ascoli theorem, the existence of a subsequence x_{n_k} such that $x_{n_k} \to x$ uniformly in $C[a,b]$. Hence H is compact.

Q.E.D.

Examples. The following equations have 2π -periodic solutions.

(a) $x''(t)+\arctan x(t)+k\arctan x(t-1) = \sin t$; k is an arbitrary small constant.

(b) $x''(t)+\arctan x(t) +K\arctan x(t-1)+\sin x(t)=\eta_1 \sin t+\eta_2 \cos t$.

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